MATH 103 Pre-Calculus Mathematics
Final Exam Reference Sheet

Chapter 1:

Circles: An equation of the form $x^2 + y^2 + ax + by = c$ describes a circle on the plane. By "completing the square", we can transform it into standard form, $(x - h)^2 + (y - k)^2 = r^2$, from which the center (h, k) and radius r are apparent.

Lines: There are three standard forms of equations describing lines:

General Linear Equation: $Ax + By + C = 0$

Point-Slope: $y - y_1 = m(x - x_1)$ describes the line with slope m passing through point (x_1, y_1)

Slope-Intercept: $y = mx + b$ describes the line with slope m passing through point $(0, b)$

Parabolas: The standard form for an equation describing a parabola is $y = ax^2 + bx + c$ $(a \neq 0)$. By completing the square, we can transform it into Vertex Form, $y = a(x - h)^2 + k$, from which the vertex (h, k) is apparent. The sign of a indicates whether the parabola opens upwards $(a > 0)$ or downwards $(a < 0)$.

Chapter 2:

Let f and g be functions mapping real numbers to real numbers. Then we have the following definitions:

$(f + g)(x) = f(x) + g(x)$ $\qquad$ $(f \cdot g)(x) = f(x) \cdot g(x)$

$(f - g)(x) = f(x) - g(x)$ $\qquad$ $(f/g)(x) = f(x)/g(x)$

Clearly, to successfully apply any of these functions to a particular value x_1, x_1 must be in the domains of both f and g.

Hence, the domain of each of $(f + g)$, $(f - g)$, and $(f \cdot g)$ excludes any value that is not in the domains of both f and g. (In other words, each one's domain is the intersection of the domains of f and g.) The domain of (f/g) further excludes any value x such that $g(x) = 0$, for reasons that should be obvious.

Function Composition: We also define $(f \circ g)$, the composition of f and g, as follows:

$$(f \circ g)(x) = f(g(x))$$

Because the application of $(f \circ g)$ to a value x_1 involves applying g to x_1 (to obtain $g(x_1)$) and then applying f to the result (to obtain $f(g(x_1))$), for x_1 to be in the domain of $(f \circ g)$ requires
(a) x_1 to be in the domain of g, and
(b) $g(x_1)$ to be in the domain of f.

If f is **one-to-one** (meaning that $f(x_1) \neq f(x_2)$ for any two (distinct) members x_1 and x_2 of f's domain), f has an **inverse function**, often denoted f^{-1}, satisfying the property that $f(x_1) = y_1$ iff $f^{-1}(y_1) = x_1$. (The domain of f is the range of f^{-1} and vice versa.)

It follows that, for all x in the domain of f, $(f^{-1} \circ f)(x) = x$, and, similarly, for all x in the domain of f^{-1}, $(f \circ f^{-1})(x) = x$.

Chapter 3:

The Factor Theorem: For a polynomial $P(x)$ and constant c, $P(c) = 0$ if and only if $x - c$ is a factor of P.

The Rational Zero Test: Let $\frac{p}{q}$ be a rational number expressed in simplest terms and let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with **integer** coefficients. If $\frac{p}{q}$ is a zero of $P(x)$, then p is a divisor of a_0 and q is a divisor of a_n.

Same Zeros Theorem: Let $f(x)$ be a function, c a non-zero constant, and $g(x) = c \cdot f(x)$. Then $f(x)$ and $g(x)$ have the same set of zeros (and each zero has the same multiplicity in f as in g).

Quadratic Formula: The zeros of $ax^2 + bx + c$ satisfy

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the *discriminant*, $b^2 - 4ac$, exceeds zero, the formula yields two simple zeros. If $b^2 - 4ac = 0$, the formula yields one zero, $x = -b/2a$, of multiplicity two. If $b^2 - 4ac < 0$, the formula yields two complex zeros, each of which is the complex conjugate of the other.

Complex Zeros of Polynomials with Real Coefficients come in Conjugate Pairs: If $P(x)$ is a polynomial with real coefficients and $a + bi$ is a zero of $P(x)$, then so is its complex conjugate $a - bi$. (Furthermore, these two zeros have the same multiplicity.)

Horizontal Asymptote Guidelines: Let $f(x) = P(x)/Q(x)$ be a rational function.
Case 1: The degree of P exceeds that of Q. Then f has no horizontal asymptote.
Case 2: The degree of Q exceeds that of P. Then f has $y = 0$ as its horizontal asymptote.
Case 3: The degree of P equals that of Q. Then f has $y = c$ as its horizontal asymptote, where c is the ratio of the leading coefficients of P and Q.

Vertical Asymptote Guidelines: Let $f(x) = P(x)/Q(x)$ be a rational function. If, after completely factoring both P and Q and cancelling any common factors, we still have $x - a$ in the denominator, then $x = a$ is a vertical asymptote of f.

Chapter 4:

Units of Measure for Angles:

$$1 \text{ degree } = \frac{\pi}{180} \text{ radians} \approx 0.01745 \text{ radians} \quad \text{and} \quad 1 \text{ radian } = \frac{180}{\pi} \text{ degrees} \approx 57.296 \text{ degrees}$$

Hence, k degrees equals $\frac{k\pi}{180}$ radians and k radians equals $\frac{180k}{\pi}$ degrees. In particular, 2π radians $= 360$ degrees (one full rotation) and π radians $= 180$ degrees (half a rotation).

A positive angle measure corresponds to a counter-clockwise rotation, a negative measure corresponds to a clockwise rotation.

Length of a Circular Arc: In a circle of radius r, the length s of the arc cut by an angle of θ radians is $s = r\theta$.

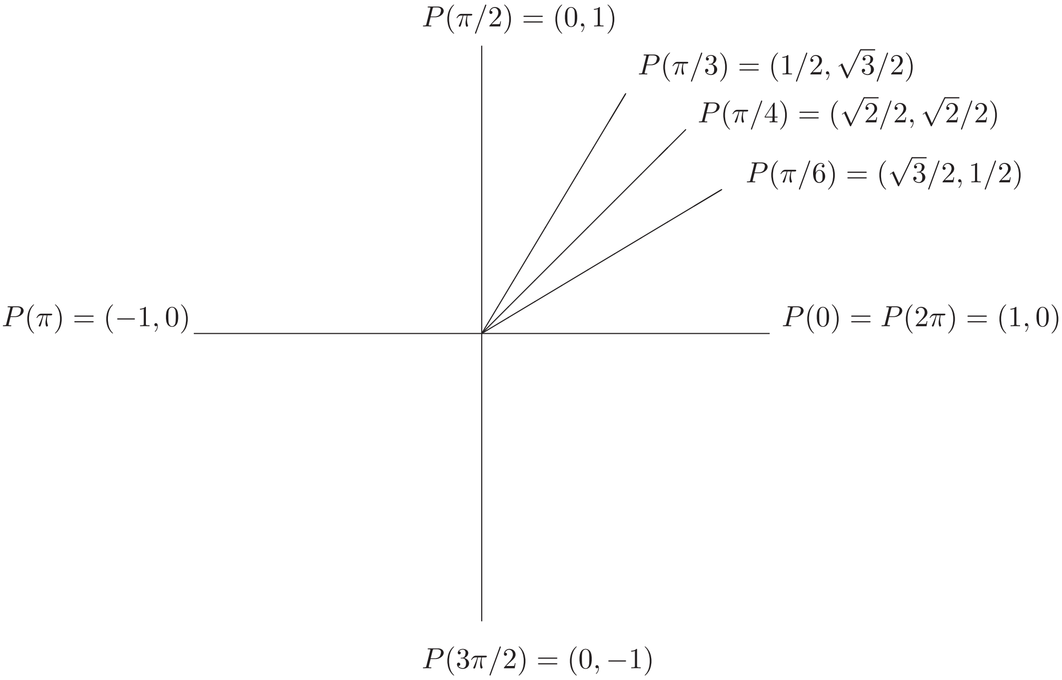
Area of a Circular Sector: In a circle of radius r, the area A of the circular sector cut by an angle of θ radians is $A = r^2\theta/2$.

Angle in Standard Position: An angle is said to be in standard position if its vertex is the origin (i.e., the point $(0, 0)$) and its initial side lies upon the x-axis where $x \geq 0$.

The Function P: We use the function P as a basis for defining the trigonometric functions sine and cosine. The domain of P is $\mathcal{R}$ (the set of real numbers) and its range is the set of points on the unit circle (i.e., the circle of radius one centered at the origin, $(0, 0)$).

Specifically, $P(t)$ is defined to be the point on the unit circle that lies on the terminal side of an angle (in standard position) of t radians. (This also corresponds to the point at which you would arrive at if you started at $(1, 0)$ and walked along the circle for $|t|$ units of distance, traveling counter-clockwise if $t > 0$ and clockwise if $t < 0$.

Here we show the value of $P(t)$ for several vital values of t. (If the unit circle is missing, you'll have to sketch it!)



Sine and Cosine Functions: We define $\sin t$ to be the y-coordinate of $P(t)$ and $\cos t$ to be the x-coordinate of $P(t)$. That is, for all t, $P(t) = (\cos t, \sin t)$.

Reference Number/Angle: For every real number t, there exists a number r_t (called t's reference number/angle) in the interval $[0, \pi/2]$ such that $P(t)$ and $P(r_t)$ are the same, except that one or both of their coordinates could be opposite in sign.

To calculate r_t, first "normalize" t by adding (or subtracting) whatever multiple of 2π puts it in the interval $[0, 2\pi)$. Suppose that $P(t) = (a, b)$. There are four cases:

$P(t)$ is in quadrant I (i.e., $0 \leq t \leq \pi/2$). Then $r_t = t$ and we have $P(r_t) = (a, b) = P(t)$.

$P(t)$ is in quadrant II (i.e., $\pi/2 \leq t \leq \pi$). Then $r_t = \pi - t$ and we have $P(r_t) = (-a, b)$.

$P(t)$ is in quadrant III (i.e., $\pi \leq t \leq 3\pi/2$). Then $r_t = t - \pi$ and we have $P(r_t) = (-a, -b)$.

$P(t)$ is in quadrant IV (i.e., $3\pi/2 \leq t < 2\pi$). Then $r_t = 2\pi - t$ and we have $P(r_t) = (a, -b)$.

Using reference numbers/angles, we can easily augment the figure above to show the values of $P(t)$, where t is any multiple of either $\pi/6$ or $\pi/4$. (Of course, this means that we also can determine the values of $\sin t$ and $\cos t$ for any such values of t.)

Graphs of Sine and Cosine:

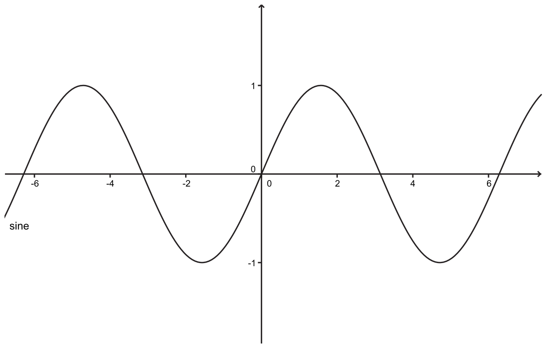


Figure 1: Graph of sine

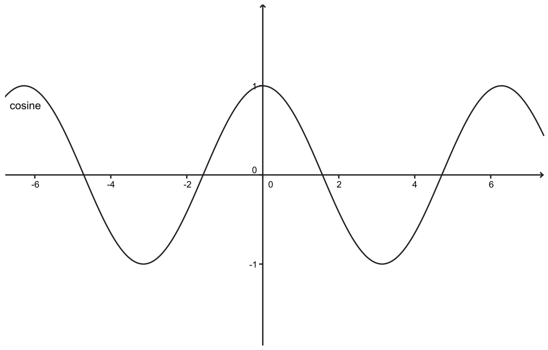


Figure 2: Graph of cosine

Notice that both sine and cosine have range $[-1, 1]$ and period 2π. Sine is odd (i.e., $\sin(-t) = -\sin t$) and cosine is even (i.e., $\cos(-t) = \cos t$).

Other Trigonometric Functions:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Trigonometric Identities:

$$\sin(x + \pi/2) = \cos x \qquad \sin(x - \pi/2) = -\cos x$$

$$\cos(x-\pi/2)=\sin x \qquad \cos(x+\pi/2)=-\sin x$$
$$\sin(x-\pi)=\sin(x+\pi)=-\sin x$$
$$\cos(x-\pi)=\cos(x+\pi)=-\cos x$$

The **Pythagorean Identity** is $\sin^2 x+\cos^2 x=1$.

The **Sum/Difference Formulas** are:
$$\begin{aligned}\sin(x_1\pm x_2) &= \sin x_1\cos x_2\pm\cos x_1\sin x_2\\ \cos(x_1+x_2) &= \cos x_1\cos x_2-\sin x_1\sin x_2\\ \cos(x_1-x_2) &= \cos x_1\cos x_2+\sin x_1\sin x_2\end{aligned}$$

From the Sum/Difference Formulas we can derive the **Double-Angle Formulas**:
$\sin 2x=2\sin x\cos x$
$\cos 2x=\cos^2 x-\sin^2 x$
$\cos 2x=2\cos^2 x-1$
$\cos 2x=1-2\sin^2 x$

Function Graphing:

Let c be a constant and suppose that we have a graph of $y=f(x)$, where f is some function.

Horizontal Shift:
To obtain a graph of $y=g(x)=f(x+c)$, shift the graph of $y=f(x)$ horizontally $|c|$ units, to the left in the case $c>0$ but to the right in the case $c<0$.

That is, for each point (x_1,y_1) on the graph of $y=f(x)$, the graph of $y=g(x)$ includes the point (x_1-c,y_1), because $g(x_1-c)=f((x_1-c)+c)=f(x_1)=y_1$.

Vertical Shift:
To obtain a graph of $y=g(x)=f(x)+c$, shift the graph of $y=f(x)$ vertically $|c|$ units, upward in the case $c>0$ but downward in the case $c<0$.

That is, for each point (x_1,y_1) on the graph of $y=f(x)$, the graph of $y=g(x)$ includes the point (x_1,y_1+c), because $g(x_1)=f(x_1)+c=y_1+c$.

Vertical Elongation/Compression:
To obtain a graph of $y=g(x)=c\cdot f(x)$, vertically elongate the graph by a factor of $|c|$ and reflect it about the x-axis if $c<0$. (If $|c|<1$, the "elongation" would actually be a compression.)

That is, for each point (x_1,y_1) on the graph of $y=f(x)$, the graph of $y=g(x)$ includes the point (x_1,cy_1), because $g(x_1)=c\cdot f(x_1)=cy_1$.

Horizontal Elongation/Compression:
To obtain a graph of $y=g(x)=f(cx)$, horizontally compress the graph by a factor of $|c|$ and, if $c<0$, reflect it about the y-axis. (If $|c|<1$, the "compression" would actually be an elongation.)

That is, for each point (x_1, y_1) on the graph of $y = f(x)$, the graph of $y = g(x)$ includes the point $(\frac{x_1}{c}, y_1)$, because $g(\frac{x_1}{c}) = f(c\frac{x_1}{c}) = f(x_1) = y_1$.

Example : Let a, b, c, and d be constants, and suppose that $g(x) = af(bx + c) + d$. How is the graph of $y = g(x)$ obtained from that of $y = f(x)$?

Solution: First observe that $bx+c = b(x+\frac{c}{b})$. Thus, we can express g by $g(x) = af(b(x+\frac{c}{b}))+d$.

Define $g_1(x) = af(x)$.
Define $g_2(x) = g_1(bx) = af(bx)$.
Define $g_3(x) = g_2(x + \frac{c}{b}) = af(b(x + \frac{c}{b}))$.
Define $g_4(x) = g_3(x) + d = af(b(x + \frac{c}{b})) + d$.

Notice that $g = g_4$!!

Starting with the graph of $y = f(x)$, we obtain the graph of $y = g_1(x) = af(x)$ by vertically elongating by a factor of $|a|$ (and reflecting about the x-axis if $a < 0$).

Then we obtain the graph of $y = g_2(x) = g_1(bx)$ by horizontally compressing the graph of $y = g_1(x)$ by a factor of $|b|$ (and reflecting about the y-axis if $b < 0$).

Then we obtain the graph of $y = g_3(x) = g_2(x + \frac{c}{b})$ by shifting the graph of $y = g_2(x)$ horizontally $|\frac{c}{b}|$ units (to the left or right, according to whether $c/b > 0$ or $c/b < 0$, respectively).

Finally, we obtain the graph of $y = g_4(x) = g_3(x) + d = g(x)$ by shifting the graph of $y = g_3(x)$ vertically $|d|$ units (upward or downward according to whether $d > 0$ or $d < 0$, respectively).