MATH 103 Pre-Calculus Mathematics
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Fall 2008 Final Exam Sample Solutions

In these solutions, FD refers to the course textbook (*PreCalculus* (4th edition), by Faires and DeFranza, published by Thomson, 2007) and RS refers to the "final exam reference sheet" to which there is a link on the course web page.

Section A

1. (a) 40 degrees equals __________ radians.
(b) For what t in the interval $[0, 2\pi)$ does the angle of t radians coincide with the angle of $-\frac{23\pi}{3}$ radians?
(c) In a particular circle, the arc cut by an angle of $\frac{2\pi}{3}$ radians has length 7. What is the circle's radius?

Solution: (a) As indicated in RS (page 2) and in FD (page 188), to convert an angle measurement in degrees to one in radians, multiply by $\pi/180$. Here we get $40\pi/180$, or $2\pi/9$.

(b) Two angles "coincide" if their measures differ by a multiple of 2π radians. (This was stated on the web page listing recommended exercises for Chapter 4.) Hence, we seek a multiple of 2π such that, by adding it to $-23\pi/3$, we obtain a value in the interval $[0, 2\pi)$. Because $-8\pi < -23\pi/3 < -7\pi$, the correct multiple of 2π is 8π. Adding 8π (which is $24\pi/3$) to $-23\pi/3$, we get $\pi/3$ as the answer.

(c) As indicated in RS (page 3) FD (page 191), the length s of the arc cut by an angle of θ radians on a circle of radius r is $s = r\theta$. Solving for r, we get $r = s/\theta$. Plugging in 7 for s and $2\pi/3$ for θ (and then doing the smallest bit of algebra), we get $\frac{21}{2\pi}$ as the answer.

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2. Suppose that $P(t) = (\frac{1}{4}, \frac{\sqrt{15}}{4})$. Determine

(a) $P(-t)$
(b) $P(t + \pi)$
(c) $P(t + \frac{\pi}{2})$ *Hint:* $\sin(x + \pi/2) = \cos x$, $\cos(x + \pi/2) = -\sin x$
(d) $P(t - 6\pi)$

Solution: Recall from RS (page 3) and FD (page 194) that, for all x, $P(x) = (\cos x, \sin x)$. Hence, we are given that $\cos t = 1/4$ and $\sin t = \sqrt{15}/4$. In particular, this means that $P(t)$ lies in quadrant I (otherwise, at least one of $\sin t$ or $\cos t$ would be negative), which is to say that $0 \leq t \leq \pi/2$.

(a)

$$\begin{aligned} P(-t) &= (\cos(-t), \sin(-t)) && \text{(property of } P\text{)} \\ &= (\cos t, -\sin t) && \text{(cos is even; sin is odd)} \\ &= (1/4, -\sqrt{15}/4) && (\cos t = 1/4; \sin t = \sqrt{15}/4) \end{aligned}$$

Alternative solution: From the discussion in FD (page 201) regarding how to compute reference numbers/angles, it is clear that, if $P(t) = (a, b)$ is in quadrant I, then $P(-t) = (a, -b)$. Here we have $a = 1/4$ and $-b = -\sqrt{15}/4$.

(b)

$$\begin{array}{rcll} P(t+\pi) & = & (\cos(t+\pi), \sin(t+\pi)) & \text{(property of } P) \\ & = & (-\cos t, -\sin t) & \text{(RS (page 5) and FD (page 208))} \\ & = & (-1/4, -\sqrt{15}/4) & (\cos t = 1/4; \sin t = \sqrt{15}/4) \end{array}$$

Alternative solution: From the discussion in FD (page 201) regarding how to compute reference numbers/angles, it is clear that, if $P(t) = (a, b)$, then $P(t + \pi) = (-a, -b)$. (In other words, $P(t)$ and $P(t + \pi)$ lie directly opposite to one another on the unit circle.) Here we have $-a = -1/4$ and $-b = -\sqrt{15}/4$.

(c)

$$\begin{array}{rcll} P(t+\pi/2) & = & (\cos(t+\pi/2), \sin(t+\pi/2)) & \text{(property of } P) \\ & = & (-\sin t, \cos t) & \text{(hint, RS (pages 4, 5), and FD (page 208))} \\ & = & (-\sqrt{15}/4, 1/4) & (\cos t = 1/4; \sin t = \sqrt{15}/4) \end{array}$$

Alternative solution: Here, arguing strictly in terms of P, we claim that, if $P(t) = (a, b)$, then either $P(t+\pi/2) = (-b, a)$ or $P(t+\pi/2) = (b, -a)$, the former in case $P(t)$ is in quadrants I or III and the latter in case $P(t)$ is in quadrants II or IV. The solution readily follows. Because this property of P is not particularly obvious, the hints were provided.

(d)

$$\begin{array}{rcll} P(t-6\pi) & = & (\cos(t-6\pi), \sin(t-6\pi)) & \text{(property of } P) \\ & = & (\cos t, \sin t) & (2\pi \text{ divides } -6\pi \text{ and } 2\pi \text{ is period of } \sin, \cos) \\ & = & (1/4, \sqrt{15}/4) & (\cos t = 1/4; \sin t = \sqrt{15}/4) \end{array}$$

Alternative solution: As noted in FD (page 196), P is periodic with period 2π, which is to say that $P(t) = P(t + 2k\pi)$ for every integer k. Taking $k = -3$, we get that $P(t) = P(t - 6\pi)$.

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3. For each given value of t, find t's reference number/angle r_t:

(a) $t = \pi/7$
(b) $t = 5\pi/4$
(c) $t = -4\pi/3$
(d) $t = 17\pi/2$

Solution: All solutions rely upon the rules regarding how to calculate r_t from t. See RS (page 4) or FD (page 201).

(a) Because $P(t)$ is in quadrant I, $r_t = t = \pi/7$.
(b) Because $P(t)$ is in quadrant III, $r_t = t - \pi = \pi/4$.
(c) First normalize t by adding 2π, which yields $t' = 2\pi/3$. Because $P(t')$ (which is also $P(t)$, of course) is in quadrant II, $r_t = r_t' = \pi - t' = \pi/3$.

(d) First normalize t by subtracting 8π, which yields $t' = \pi/2$. Because $P(t')$ (which is also $P(t)$, of course) is in quadrant I (on border between quadrants I and II?), $r_t = r'_t = t' = \pi/2$.

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4. For each given value of t, find both $\sin t$ and $\cos t$:

Solution: Because $P(x) = (\cos x, \sin x)$ for all x, to answer the question it suffices to determine $P(t)$ for each given value of t. To accomplish this, we

find the reference angle r_t of t (as in the solution to the previous problem),

determine $P(r_t)$ (using the given values of P in RS (page 3)), and

obtain $P(t)$ by adjusting the signs of the coordinates of $P(r_t)$ according to the quadrant in which $P(t)$ lies.

(a) $t = -4\pi/3$
Solution: First normalize t by adding 2π and thereby obtaining $t' = 2\pi/3$. Because $P(t)$ is in quadrant II, its reference angle is $r_t = \pi - t' = \pi/3$. We have $P(\pi/3) = (1/2, \sqrt{3}/2)$. As $P(t)$ is in quadrant II, we must flip the sign of the x-coordinate, thereby obtaining $P(t) = (-1/2, \sqrt{3}/2)$.

(b) $t = 7\pi/6$
Solution: Because $P(t)$ is in quadrant III, its reference angle is $r_t = t - \pi = \pi/6$. We have $P(\pi/6) = (\sqrt{3}/2, 1/2)$. As $P(t)$ is in quadrant III, we must flip the signs of both coordinates, thereby obtaining $P(t) = (-\sqrt{3}/2, -1/2)$.

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5. For each equation, identify all t in $[0, 2\pi)$ that satisfy it.

(a) $\cos t = -\frac{1}{2}$
Solution: On RS (page 3) we find that $\cos(\pi/3) = 1/2$. Hence, any t in $[0, 2\pi)$ for which $r_t = \pi/3$ (corresponding to values of t for which $|\cos t| = 1/2$) and $\cos t < 0$ (corresponding to values of t for which $P(t)$ is in either quadrant II or quadrant III) is a solution.

For t to be in the desired interval with $P(t)$ being in quadrant II, it must be that $\pi/2 \leq t \leq \pi$, in which case we have (according to RS (page 4) and FD (page 201)) $r_t = \pi - t$. Plugging in $\pi/3$ for r_t and solving for t, we get $t = 2\pi/3$.

For t to be in the desired interval with $P(t)$ being in quadrant III, it must be that $\pi \leq t \leq 3\pi/2$, in which case we have (according to RS (page 4) and FD (page 201)) $r_t = t - \pi$. Plugging in $\pi/3$ for r_t and solving for t, we get $t = 4\pi/3$.

The two solutions are therefore $t = 2\pi/3$ and $t = 4\pi/3$.

(b) $\sin t = \frac{1}{2}$
Solution: Following reasoning similar to that used in the solution to (a), we obtain as solutions $t = \pi/6$ and $t = 5\pi/6$.

(c) $\sin 2t = 1$
Solution: If $\sin x = 1$, then, letting $t = x/2$, we have

$$\sin 2t = \sin(2(x/2)) = \sin x = 1$$

If t is in the interval $[0, 2\pi)$, then x (which is $2t$) is in the interval $[0, 4\pi)$. Hence, we must find all x in $[0, 4\pi)$ satisfying $\sin x = 1$ and, for each one, divide it by two in order to arrive at a solution (for t).

The values of x in $[0, 4\pi)$ satisfying $\sin x = 1$ are $x = \pi/2$ and $\pi/2 + 2\pi$. Recalling that $t = x/2$, this corresponds to $t = \pi/4$ and $t = \pi/4 + \pi = 5\pi/4$.

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6. Find all t in $[0, 2\pi)$ satisfying $2\sin^2 t - \sin t - 1 = 0$. (*Hint:* Factor it.)

Solution:

$$\begin{array}{lll} & 2\sin^2 t - \sin t - 1 = 0 & \\ \equiv & (2\sin t + 1)(\sin t - 1) = 0 & \text{(factoring)} \\ \equiv & 2\sin t + 1 = 0 \;\text{ or }\; \sin t - 1 = 0 & (a \cdot b = 0 \text{ iff } a = 0 \text{ or } b = 0) \\ \equiv & \sin t = -1/2 \;\text{ or }\; \sin t = 1 & \text{(algebra)} \end{array}$$

Using the same approach as in the solution to the previous problem, we find that $\sin t = -1/2$ has as solutions (within the specified interval) $t = 7\pi/6$ and $t = 11\pi/6$ (namely, each angle having terminal side in quadrant III or IV and having $\pi/6$ as its reference angle). Also, $\sin t = 1$ has as its solution (within the specified interval) $t = \pi/2$.

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7. Suppose that $\sin t = \frac{2}{3}$. Then what are the possible values of $\cos t$?

Solution:

$$\begin{array}{lll} & \sin^2 t + \cos^2 t = 1 & \text{(Pythagorean Identity)} \\ \equiv & \cos^2 t = 1 - \sin^2 t & \text{(algebra)} \\ \equiv & \cos^2 t = 1 - (2/3)^2 & \text{(assumption } \sin t = 2/3) \\ \equiv & \cos^2 t = 5/9 & \text{(arithmetic)} \\ \equiv & \cos t = \pm\sqrt{5/9} & \text{(algebra)} \\ \equiv & \cos t = \pm\sqrt{5}/3 & (\sqrt{a/b} = \sqrt{a}/\sqrt{b};\, \sqrt{9} = 3) \end{array}$$

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8. Use one of the Sum/Difference Formulas to calculate $\cos\frac{\pi}{12}$. (Note that $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$).

Solution:

$$\begin{array}{lll} \cos\frac{\pi}{12} & = \cos(\frac{\pi}{3} - \frac{\pi}{4}) & (1/12 = 1/3 - 1/4) \\ & = \cos\frac{\pi}{3} \cdot \cos\frac{\pi}{4} + \sin\frac{\pi}{3} \cdot \sin\frac{\pi}{4} & \text{(Difference Formula)} \\ & = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} & \text{(using } P(t) \text{ values in RS)} \\ & = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}\sqrt{3}}{4} & \text{(arithmetic)} \\ & = \frac{\sqrt{2}(1+\sqrt{3})}{4} & \text{(arithmetic)} \end{array}$$

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9. Use a Double-Angle Formula in order to calculate $\sin \frac{\pi}{2}$.

Solution:

$$\begin{aligned}
\sin \tfrac{\pi}{2} &= \sin(2 \cdot \tfrac{\pi}{4}) && \text{(arithmetic)} \\
&= 2 \cdot \sin \tfrac{\pi}{4} \cdot \cos \tfrac{\pi}{4} && \text{(Double} - \text{Angle Formula)} \\
&= 2 \cdot \tfrac{\sqrt{2}}{2} \cdot \tfrac{\sqrt{2}}{2} && (\sin \pi/4 = \cos \pi/4 = \sqrt{2}/2) \\
&= 1 && \text{(arithmetic)}
\end{aligned}$$

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10. Chris needs to look up at a 45-degree angle in order to see the top edge of a building that is 65-feet tall. How far from the building is Chris standing? (For a little extra credit, make your calculation account for the fact that Chris's eyes are five feet above the ground.)

Solution: The nearby figure depicts the situation. The vertical line segment (of length 65) corresponds to the side of the building, and the horizontal line segment (of unknown length x) runs from Chris's feet to where the building meets the ground. The third line segment, which completes the right triangle, is Chris's line of sight to the top of the building. It forms a 45-degree angle with the horizontal line segment.

Recall that, in a right triangle, the tangent of an angle is the ratio of the lengths of the opposite and adjacent sides ("TOA"). Thus, we have $\tan 45° = 65/x$, or $x = 65/\tan 45°$. But

$$\tan 45° = \tan \frac{\pi}{4} = \sin \frac{\pi}{4} / \cos \frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

It follows that $x = 65$.

For the extra credit part, we interpret the horizontal line segment as running from Chris's eyes to a point on the side of the building five feet above the ground. Under this interpretation, the vertical line segment's length is only 60. Solving the problem in exactly the same way as before, we get an answer of $x = 60$.

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      /|
     / |
    /  |
   /   | 65
  /    |
 /45   |
+------+
   x
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Figure 1: Drawing for A.10.

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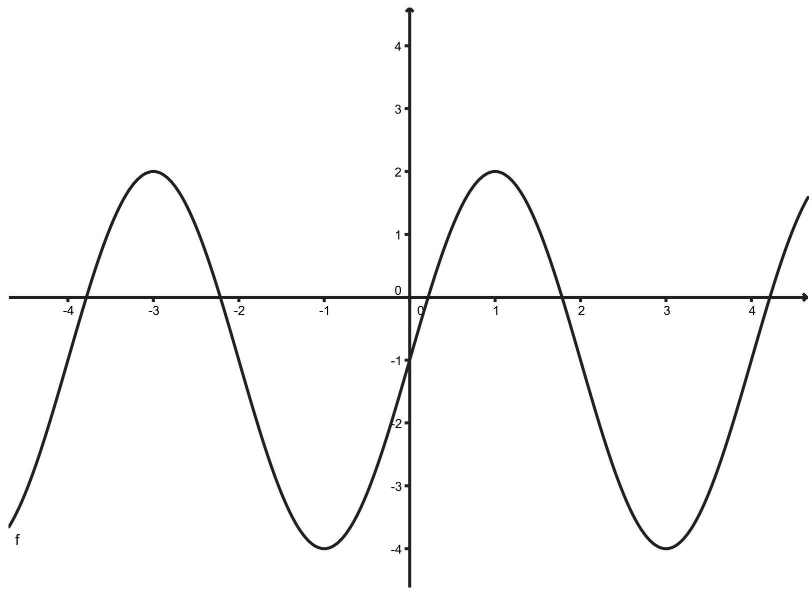
11. For some constants A, B, C, and D, the graph below is that of $y = A\cos(Bx + C) + D$. Identify A, B, C, and D.
Hint: Use the difference between the maximum and minimum values to determine A, use the period to determine B, use the values of x at which minima and maxima occur (relative to

where they occur in the graph of $y = \cos x$) to determine C/B, and use maximum and minimum values (relative to those of $\cos x$, and taking into account A's influence) to determine D.



Solution: For convenience, we refer to the function depicted in the graph as f.

The distance between extrema in f is 6 (from a minimum of -4 to a maximum of 2), whereas it is only 2 in the cosine function (from a minimum of -1 to a maximum of 1). Hence, the vertical elongation factor, A, is 3 (i.e., 6/2).

The period of f is 4 (notice, for example, that the maxima depicted are at $x = -3$ and $x = 1$), whereas the period of cosine is 2π. Recall that, if g is periodic with period T, then $h(x) = g(cx)$ has period T/c. (This fact appears in the web page listing recommended exercises for Chapter 4 and also —in a statement specifically pertaining to the sine and cosine functions— in FD (page 211).) So the horizontal compression factor, B, satisfies $4 = 2\pi/B$, which works out to $B = \pi/2$.

So far, we think that $A = 3$ and $B = \pi/2$. Compare the graph of $y = g(x) = A\cos(Bx) = 3\cos(\frac{\pi}{2}x)$ (which we get from the graph of $y = \cos x$ by vertically elongating by a factor of three and horizontally compressing by a factor of $\pi/2$) with that of $y = f(x)$ (i.e., the given graph).

The two graphs are in the nearby figure, with g shown using a dashed line. To get from g to f, we need to shift one unit to the right and one unit down. Hence, $f(x) = g(x-1) - 1$. Using this, we can derive the proper choices for C and D:

$$\begin{array}{rcll} f(x) & = & g(x-1) - 1 & \text{(observation from above)} \\ & = & 3\cos(\frac{\pi}{2}(x-1)) - 1 & \text{(definition of } g\text{)} \\ & = & 3\cos(\frac{\pi}{2}x - \frac{\pi}{2})) - 1 & \text{(algebra)} \\ & = & A\cos(Bx + C) + D & \text{(choosing } A = 3, B = \frac{\pi}{2}, C = -\frac{\pi}{2}, D = -1\text{)} \end{array}$$

The answer, then, is $A = 3$, $B = \frac{\pi}{2}$, $C = -\frac{\pi}{2}$, and $D = -1$.

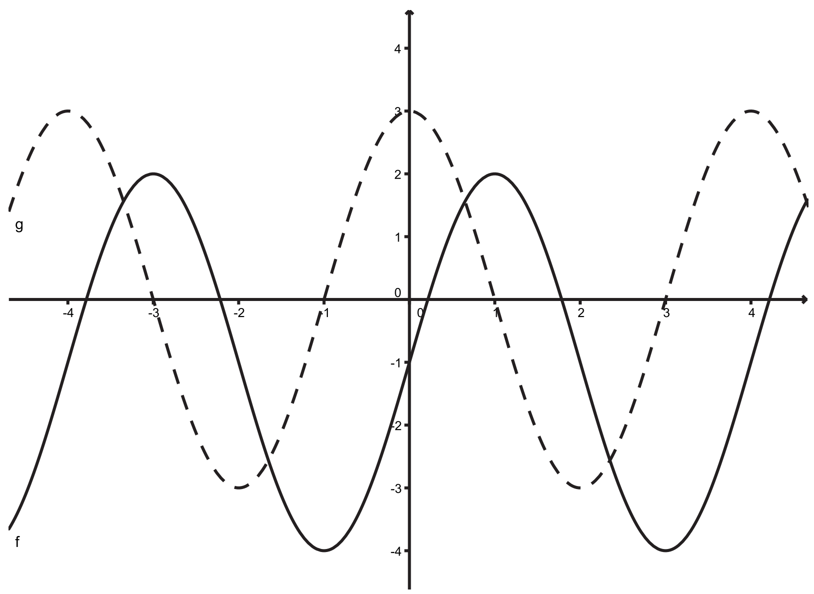


Figure 2: Graphs for A.11.

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12. If we restrict the domain of the sine function to the interval $[-\pi/2, \pi/2]$, we get a one-to-one function. Call it $\sin_1$. Being one-to-one, $\sin_1$ has an inverse function, which is called arcsin.

(a) What is $\arcsin(\sin(\pi/4))$?
Solution: $\pi/4$. (Recall that $f^{-1}(f(x)) = x$.)

(b) What is $\arcsin(1/2)$?
Solution: The answer is that value t in the interval $[-\pi/2, \pi/2]$ satisfying $\sin t = 1/2$. On RS (page 3), we find that the y-coordinate of $P(\pi/6)$ is $1/2$, which is to say that $\sin \pi/6 = 1/2$. Hence, the answer is $\pi/6$.

Section B

1. Give the standard form of the equation of the circle described by $x^2 - 2x + y^2 + 4y = 11$, identify its center and radius, and sketch it in the space below.

Solution:

$$\begin{aligned} & x^2 - 2x + y^2 + 4y = 11 \\ \equiv\ & (x^2 - 2x + 1) + (y^2 + 4y + 4) = 11 + 1 + 4 \quad \text{(completing the square, step 1)} \\ \equiv\ & (x - 1)^2 + (y + 2)^2 = 16 \quad \text{(completing the square, step 2)} \end{aligned}$$

The last equation above is in standard form with $h = 1$, $k = -2$, and $r = \sqrt{16} = 4$, which tells us that the circle's center is at $(1, -2)$ and that its radius is 4.

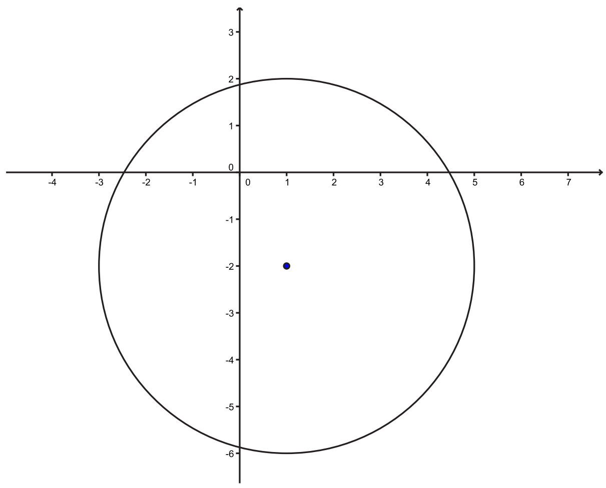

Figure 3: Graph for B.1.

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2. Find an equation (in any of the standard forms) describing the line that passes through $(4, 3)$ and is parallel to the line given by $2x - 4y + 3 = 0$

Solution: Recall that the slope of the line expressed by $Ax + By + C = 0$ (a general linear equation) is $-\frac{A}{B}$. The given line is expressed this way, with $A = 2$ and $B = -4$. Hence, its slope is $-\frac{2}{-4}$, or $\frac{1}{2}$. The line we seek is parallel to this, which means that its slope is $\frac{1}{2}$, too.

A line with slope m passing through a point (x_1, y_1) is given by the equation (in point-slope form) $y - y_1 = m(x - x_1)$. The line we seek has slope $\frac{1}{2}$ (as indicated above) and passes through $(4, 3)$. Hence, it is described by the equation $y - 3 = \frac{1}{2}(x - 4)$ (in point-slope form).

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3. Let $f(x) = \sqrt{2x} - 1$ and $g(x) = x - 5$.
(a) Describe $(f \circ g)(x)$ and identify its domain and range.

Solution:

$$\begin{array}{rcll} (f \circ g)(x) & = & f(g(x)) & \text{(definition of } f \circ g) \\ & = & f(x-5) & \text{(definition of } g) \\ & = & \sqrt{2(x-5)} - 1 & \text{(definition of } f, \text{ function application)} \\ & = & \sqrt{2x-10} - 1 & \text{(algebra)} \end{array}$$

The domain includes all x for which $2x - 10 \geq 0$, which corresponds to $x \geq 5$ (or, as an interval, $[5, \infty)$). The range includes every value that can be produced by the expression $\sqrt{2x-10} - 1$, which is the interval $[-1, \infty)$.

(b) Describe $(g \circ f)(x)$ and identify its domain and range.
Solution:

$$\begin{array}{rcll} (g \circ f)(x) & = & g(f(x)) & \text{(definition of } g \circ f) \\ & = & g(\sqrt{2x-1}) & \text{(definition of } f) \\ & = & \sqrt{2x-1} - 5 & \text{(definition of } g, \text{ function application)} \end{array}$$

The domain includes all x for which $2x - 1 \geq 0$, which corresponds to $x \geq 1/2$ (or, as an interval, $[1/2, \infty)$). The range includes every value that can be produced by the expression $\sqrt{2x-1} - 5$, which is the interval $[-5, \infty)$.

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4. Describe the inverse function of $f(x) = 4x^3 - 2$.

Solution: Starting with $y = 4x^3 - 2$, we swap instances of x and y and then solve for y:

$$\begin{array}{rl} & x = 4y^3 - 2 \\ \equiv & x + 2 = 4y^3 \\ \equiv & (x+2)/4 = y^3 \\ \equiv & \sqrt[3]{(x+2)/4} = y \end{array}$$

The inverse of f is therefore $f^{-1}(x) = \sqrt[3]{(x+2)/4}$

An alternative approach is to observe that, to apply f, we take the input and (1) cube it, (2) multiply by 4, and (3) subtract 2. If we invert each step and do them in reverse order, the resulting sequence of steps is (1) add 2, (2) divide by 4, and (3) take the cube root. Assuming that the input is called x, the corresponding expression is $\sqrt[3]{(x+2)/4}$, which agrees with our first answer.

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5. Describe a rational function having
(i) $y = -3$ as its horizontal asymptote,
(ii) $x = 5, -8$ as its zeros,
(iii) a "hole" at $x = 6$, and
(iv) $x = -2$ and $x = -4$ as its vertical asymptotes.

Solution:

$$f(x) = \frac{-3(x-5)(x+8)(x-6)}{(x+2)(x+4)(x-6)}$$

6. Factor completely the polynomial $P(x) = x^3 - 3x + 2$ and identify its zeros.

Solution: The Rational Zero Test tells us that the only possible rational zeros of P are $\pm 1, \pm 2$. By evaluating P at 1, we find that $P(1) = 0$, which, by the Factor Theorem, means that $x - 1$ is a factor of P. Dividing $x - 1$ into P (work not shown here), we get as a quotient $x^2 + x - 2$, which means that $P(x) = (x-1)(x^2 + x - 2)$. Factoring the quadratic, we get $P(x) = (x-1)(x-1)(x+2) = (x-1)^2(x+2)$, the zeros of which are $x = 1, -2$.

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7. Consider the parabola described by the equation $y = -x^2 + 2x + 1$. Determine its Vertex Form equation, sketch its graph in the space below, and identify the domain and range of the function described by this equation.

Solution:

$$\begin{array}{lll} & y = -x^2 + 2x + 1 & \\ \equiv & y = -(x^2 - 2x) + 1 & \text{(factor out } -1) \\ \equiv & y = -(x^2 - 2x + 1 - 1) + 1 & \text{(completing the square, step 1)} \\ \equiv & y = -(x^2 - 2x + 1) + 1 + 1 & \text{(algebra)} \\ \equiv & y = -(x-1)^2 + 2 & \text{(completing the square, step 2)} \end{array}$$

The last equation above is in vertex form with $a = -1$, $h = 1$, and $k = 2$, which tells us that the parabola's vertex is at $(1, 2)$ and that it opens downward. To ensure that the graph has some semblance of accuracy, we should at least determine the x-intercepts. Here is that calculation:

$$\begin{array}{lll} & -(x-1)^2 + 2 = 0 & \\ \equiv & (x-1)^2 = 2 & \text{(algebra)} \\ \equiv & x - 1 = \pm\sqrt{2} & \text{(algebra)} \\ \equiv & x = 1 \pm \sqrt{2} & \text{(algebra)} \end{array}$$

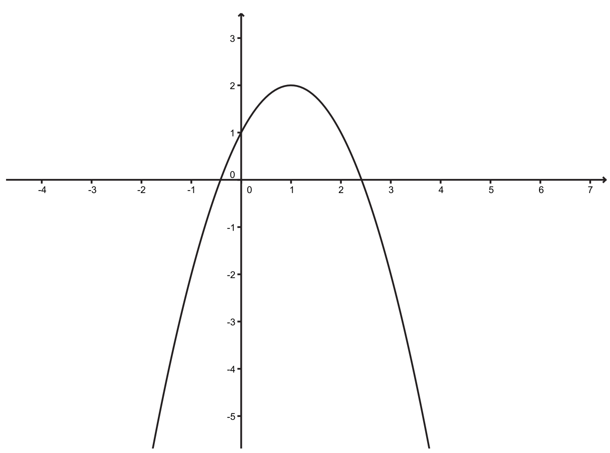

Figure 4: Graph for B.7.