MATH 103 Pre-Calculus Mathematics
Quiz #6 Fall 2008
Sample Solutions

1. Let $f(x) = x^2 + 2$ and $g(x) = \sqrt{x-5}$.

(a) Calculate $(f \circ g)(8)$.
(b) Calculate $(g \circ f)(8)$.
(c) Find $(f \circ g)$ and give its domain and range.
(d) Find $(g \circ f)$ and give its domain and range.

Recall that, for $a \geq 0$, $(\sqrt{a})^2 = a$.

Solution:
(a)

$$\begin{aligned}
(f \circ g)(8) &= f(g(8)) && \text{(by definition of } (f \circ g)\text{)} \\
&= f(\sqrt{8-5}) && \text{(by definition of } g\text{)} \\
&= f(\sqrt{3}) && \text{(by arithmetic)} \\
&= (\sqrt{3})^2 + 2 && \text{(by definition of } f\text{)} \\
&= 3 + 2 && \text{(by } a \geq 0 \text{ implies } (\sqrt{a})^2 = a\text{)} \\
&= 5 && \text{(by arithmetic)}
\end{aligned}$$

(b)

$$\begin{aligned}
(g \circ f)(8) &= g(f(8)) && \text{(by definition of } (g \circ f)\text{)} \\
&= g(8^2 + 2) && \text{(by definition of } f\text{)} \\
&= g(66) && \text{(by arithmetic)} \\
&= \sqrt{66-5} && \text{(by definition of } g\text{)} \\
&= \sqrt{61} && \text{(by arithmetic)}
\end{aligned}$$

(c) Repeating what was done in solving (a), but using x rather than 8:

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{(by definition of } (f \circ g)\text{)} \\
&= f(\sqrt{x-5}) && \text{(by definition of } g\text{)} \\
&= (\sqrt{x-5})^2 + 2 && \text{(by definition of } f\text{)} \\
&= x - 5 + 2 && \text{(by } a \geq 0 \text{ implies } (\sqrt{a})^2 = a\text{)} \\
&= x - 3 && \text{(by arithmetic)}
\end{aligned}$$

Recall that a value x is in the domain of $(f \circ g)$ if and only if it is in the domain of g and, additionally, $g(x)$ is in the domain of f. Here, the domain of g is the interval $[5, \infty)$. Because the domain of f is all of $\mathcal{R}$, it imposes no further restrictions upon the domain of $(f \circ g)$. Hence, the domain of $(f \circ g)$ is the same as g's, the interval $[5, \infty)$.

To obtain the range of $(f \circ g)$ we ask, "What values can be produced by the expression $x - 3$ if x is allowed to be any value that is 5 or greater (i.e., any value in $[5, \infty)$)?" The answer, of course, is any value that is 2 or greater (i.e., any value in $[2, \infty)$). Hence, the range of $(f \circ g)$ is $[2, \infty)$.

(d) Repeating what was done in solving (b), but using x rather than 8:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{(by definition of } (g \circ f)\text{)} \\ &= g(x^2+2) && \text{(by definition of } f\text{)} \\ &= \sqrt{(x^2+2)-5} && \text{(by definition of } g\text{)} \\ &= \sqrt{x^2-3} && \text{(by arithmetic)}\end{aligned}$$

Recall that a value x is in the domain of $(g \circ f)$ if and only if it is in the domain of f and, additionally, $f(x)$ is in the domain of g. Because f's domain is all of $\mathcal{R}$, no restrictions upon the domain of $(g \circ f)$ are thereby imposed.

As for which values of x put $f(x)$ in the domain of g, we can get that information by examining our final expression describing $(g \circ f)$, which is $\sqrt{x^2-3}$. For this to be defined requires $x^2 - 3 \geq 0$, which is to say that $x^2 \geq 3$, which is to say that either $x \leq -\sqrt{3}$ or $x \geq \sqrt{3}$. Hence, we conclude that the domain of $(g \circ f)$ is $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$.

To be a little more rigorous, we could have reasoned as follows:

$$\begin{aligned}
& f(x) \in \text{domain of } g && \\
\equiv\ & f(x) \in [5, \infty) && \text{(domain of } g \text{ is obviously } [5,\infty)\text{)} \\
\equiv\ & f(x) \geq 5 && (a \in [b,\infty) \equiv a \geq b) \\
\equiv\ & x^2 + 2 \geq 5 && \text{(definition of } f\text{)} \\
\equiv\ & x^2 - 3 \geq 0 && \text{(algebra)} \\
\equiv\ & x^2 \geq 3 && \text{(algebra)} \\
\equiv\ & |x| \geq \sqrt{3} && (b \geq 0 \text{ implies } a^2 \geq b \equiv |a| \geq \sqrt{b}) \\
\equiv\ & x \leq -\sqrt{3} \ \vee\ x \geq \sqrt{3} && (|x| \geq b \ \equiv\ (x \geq b \ \vee\ x \leq -b)) \\
\equiv\ & x \in (-\infty, -\sqrt{3}] \vee x \in [\sqrt{3}, \infty) && (a \in [-\infty, b] \equiv a \leq b;\ a \in [b, \infty) \equiv a \geq b) \\
\equiv\ & x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty) && ((a \in A \vee a \in B) \equiv (a \in A \cup B))
\end{aligned}$$

To obtain the range of $(g \circ f)$ we ask, "What values can be produced by the expression $\sqrt{x^2-3}$ if x is allowed to be any value less than or equal to $-\sqrt{3}$ or any value greater than or equal to $\sqrt{3}$?" The answer is: all nonnegative real numbers, i.e., the interval $[0, \infty)$.

To verify this rigorously, let b be any nonnegative real number. If we choose a to be $\sqrt{b^2+3}$, we have

$$(g \circ f)(a) = (g \circ f)(\sqrt{b^2+3}) = \sqrt{(\sqrt{b^2+3})^2 - 3} = \sqrt{b^2+3-3} = \sqrt{b^2} = b$$

Hence, b is in the range of $(g \circ f)$. As b can be any nonnegative real, this means that the range of $(g \circ f)$ includes all such numbers. Clearly the range includes no negative numbers and so it is precisely $[0, \infty)$.

2. Let $f(x) = 2x^3 - 7$.

(a) Assuming that f has an inverse function f^{-1}, what is $f^{-1}(9)$?
(b) Find f^{-1}.

Recall that $a^3 = b$ is equivalent to $a = \sqrt[3]{b}$.

Solution:
(a) Recall that $f(a) = b$ is equivalent to $f^{-1}(b) = a$. Hence, to determine $f^{-1}(9)$ it suffices to find a such that $f(a) = 9$. To do this, we solve the equation $9 = f(x) = 2x^3 - 7$, from which we derive $x^3 = 8$, or $x = 2$.

(b) Following the advice of Faires and DeFranza, we take the equation $y = 2x^3 - 7$ and solve for x. Then we replace x by y and vice versa to obtain an algebraic expression for f^{-1} in terms of x:

$$\begin{array}{lcll} y & = & 2x^3 - 7 & \\ y + 7 & = & 2x^3 & \text{(add 7 to both sides)} \\ (y+7)/2 & = & x^3 & \text{(divide both sides by 2)} \\ \sqrt[3]{(y+7)/2} & = & x & \text{(take cube root of each side)} \end{array}$$

Swapping the roles of x and y, we get

$$f^{-1}(x) = y = \sqrt[3]{(x+7)/2}$$

As an alternative approach, we could have observed that, in applying f, we first cube, then double, and then subtract 7. To invert that, we would add 7, then halve, and then take the cube root. Applying this sequence of operations to input x corresponds to computing $\sqrt[3]{(x+7)/2}$, which agrees with our earlier solution.

To verify that our answer is, indeed, correct, we check whether $(f^{-1} \circ f)(x) = x$:

$$\begin{array}{rll} (f^{-1} \circ f)(x) = & f^{-1}(f(x)) & \text{(definition of function composition)} \\ = & f^{-1}(2x^3 - 7) & \text{(definition of } f\text{)} \\ = & \sqrt[3]{((2x^3-7)+7)/2} & \text{(definition of } f^{-1}\text{)} \\ = & \sqrt[3]{2x^3/2} & \text{(algebra : the 7's cancel)} \\ = & \sqrt[3]{x^3} & \text{(algebra : the 2's cancel)} \\ = & x & (\sqrt[3]{a^3} = a) \end{array}$$