MATH 103 Pre-Calculus Mathematics
Test #3 Fall 2008
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Sample Solutions

1. Let $P(x) = 3x^4 + 2x^3 - x + 2$ and $D(x) = x^2 + 2x - 1$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x) = Q(x) \cdot D(x) \; + \; R(x)$. (That is, divide $D(x)$ into $P(x)$ to find the quotient $Q(x)$ and remainder $R(x)$.)

Solution: $Q(x) = 3x^2 - 4x + 11$ and $R(x) = -27x + 13$.

2. Demonstrate, in as simple a way as possible, that $x - 3$ is not a factor of $P(x) = 2x^2 - x + 4$.

Solution: By the **Factor Theorem**, $x - 3$ is a factor of P if and only if $P(3) = 0$. But evaluating P at 3 yields $P(3) = 19$. Hence, $x - 3$ is not a factor of P.

3. Let $P(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x + \frac{1}{4}$. Use the *Rational Zero Test* to compile a set of rational numbers that are candidates for being zeros of $P(x)$.

Solution: The Rational Zero Test applies to polynomials with integer coefficients. Hence, we multiply P by the least common multiple of the denominators of its coefficients in order to obtain a polynomial with integer coefficients but having the same zeros as P. (See **Same Zeros** theorem on reference sheet.) The least common multiple of 3, 2, 1, and 4 is 12. Now, $Q(x) = 12 \cdot P(x) = 4x^3 - 6x^2 + 48x + 3$. Applying the Rational Zero Test to Q, we find that there are twelve rational zero candidates: ± 1, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, ± 3, $\pm\frac{3}{2}$, and $\pm\frac{3}{4}$.

4. Determine a polynomial $P(x)$ with zeros at $x = 2$ and $x = -1$ and satisfying $P(3) = 8$.

Solution: For P to have zeros at $x = 2$ and $x = -1$ implies (by the Factor Theorem) that $x-2$ and $x + 1$ are factors of P. Hence, our first "guess" at a solution is $(x - 2)(x + 1)$. Evaluating this at $x = 3$ yields 4, which is half of what we desire. Multiplying by 2 does not change the zeros (recall the **Same Zeros** theorem); doing so yields $P(x) = 2(x - 2)(x - 1)$, which has all three properties required.

5. Determine a rational function f having a vertical asymptote at $x = -2$ (and nowhere else), zeros at $x = 4$ and $x = -3$ (and nowhere else), a horizontal asymptote at $y = 0$, and a "hole" at $x = 5$.

Solution: To get the vertical asymptote at $x = -2$, we put the factor $x+2$ in the denominator. To get zeros at $x = 4$ and $x = -3$, we put the factors $x - 4$ and $x + 3$ in the numerator. To get a "hole" at $x = 5$, we put the factor $x - 5$ into both numerator and denominator. This gives us, as a first "guess" at a solution, the rational function

$$\frac{(x-4)(x+3)(x-5)}{(x+2)(x-5)}$$

Unfortunately, this function fails to have a horizontal asymptote at $y = 0$. To achieve that, we need the denominator to have greater degree than the numerator. At this point, the numerator has degree three and the denominator has degree two, so we must increase the latter by (at least) two. At the same time, we must not introduce any "new" vertical asymptotes (such as would happen if we inserted, say, x as a factor in the denominator), nor may we lose any zeros (such as would happen if we inserted, say, $x - 4$ as a factor in the denominator).

But if we simply insert two more $x + 2$ factors into the denominator, we get no new vertical asymptotes, lose no zeros, and get the horizontal asymptote we want. So a solution is

$$f(x) = \frac{(x-4)(x+3)(x-5)}{(x+2)^3(x-5)}$$

As an alternative, we could have inserted as a factor in the denominator any polynomial of degree two (or more) having no (real) zeros! For example, $x^2 + a$, for any $a > 0$, would work.

6. Factor completely the polynomial $P(x) = x^4 - x^3 - 5x^2 + 3x + 6$.

Hint: One of its zeros is $x = -1$.

Solution: From the hint, and the Factor Theorem, we know that $x+1$ is a factor of P. Hence, we divide $x+1$ into P (work not shown here), which yields as a quotient $Q(x) = x^3-2x^2-3x+6$.

So we have that $P(x) = (x + 1) \cdot Q(x)$. It remains to factor Q. Applying the Rational Zero Test, we determine that any rational factor of Q must be a divisor of 6. After trying a few of them, we stumble upon the fact that $Q(2) = 0$, which means (by the Factor Theorem) that $x - 2$ is a factor of Q. Dividing $x - 2$ into Q (work not shown here) yields $Q_1(x) = x^2 - 3$.

So we have that $P(x) = (x + 1)(x - 2) \cdot Q_1(x)$, and it remains to factor Q_1. Our knowledge of factoring (see sample solutions for Quiz #8) tells us that $Q_1(x) = x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$. (Applying the Quadratic Formula would have yielded the same information, of course.)

The final answer, then, is $P(x) = (x + 1)(x - 2)(x - \sqrt{3})(x + \sqrt{3})$.

7. Express the polynomial $P(x) = x^3 - 5x^2 + 17x - 13$ as a product of three linear factors.

Hint: P has only one real zero.

Solution: Application of the Rational Zero Test tells us that any rational zero of P is a divisor of 13. That leads us to evaluate P at 1 and to find that $P(1) = 0$, which tells us that $x - 1$ is a factor of P. Dividing $x - 1$ into P (work not shown here), we get $Q(x) = x^2 - 4x + 13$.

So we have that $P(x) = (x - 1) \cdot Q(x)$. It remains to factor Q. The hint (togeher with the easily-verified fact that 1 is not a zero of Q) tells us that Q has no real zeros (otherwise, P would have more than one, in contradiction to the hint), so there is no sense in applying the Rational Zero Test. Rather, we use the Quadratic Formula to find that the zeros of Q satisfy

$$x = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6\sqrt{-1}}{2} = 2 \pm 3i$$

The desired factorization of P is thus $P(x) = (x - 1)(x - (2 + 3i))(x - (2 - 3i))$.

8. Let

$$f(x) = \frac{(x + 2)(x - 1)}{(x + 2)(x - 3)}$$

Identify the domain and range of f, as well as any horizontal asymptotes, vertical asymptotes, "holes", and zeros.

In the space below, sketch the graph of f (and make sure that it is consistent with the first part of your answer).

Solution: The $x + 2$ factors in the numerator and denominator tell us that f has a "hole" at $x = -2$. By cancelling those factors, we get

$$f(x) = \frac{x - 1}{x - 3} \quad (\text{for } x \neq -2)$$

from which we gather that f's lone vertical asymptote is $x = 3$ and its lone zero is $x = 1$. The domain of f includes all real numbers, except where it has a hole or vertical asymptote, which is to say $\mathcal{R} - \{-2, 3\}$ or, in terms of intervals, $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

Because the numerator and denominator of f are of the same degree and the ratio of their leading coefficients is 1, f has $y = 1$ as its horizontal aymptote.

To obtain the range of f, it helps to sketch the graph of f. Because $x = 3$ is a vertical asymptote, we know that, as x approaches 3 from the left, and similarly from the right, $f(x)$ approaches either $+\infty$ or $-\infty$.

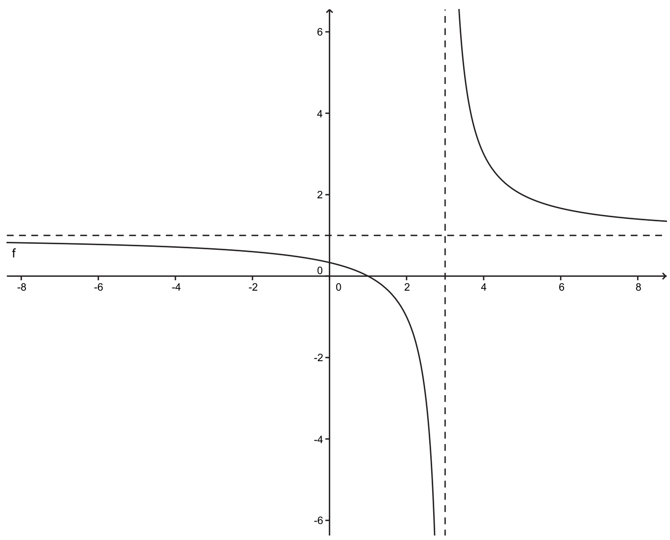
Consider values of x slightly larger (i.e., "to the right of") 3. There, the value of f is a large positive number. (For example, $f(3.001) = 2.001/0.001 = 2001$.) As x increases from 3, f gets smaller and smaller. For example, $f(4) = 3$, $f(5) = 2$, $f(7) = 1.5$, $f(100) = 99/97 \approx 1.02$, $f(100) = 999/997 \approx 1.002$, etc. Indeed, as $x > 3$ increases, $f(x)$ will approach 1 from "above", but never quite reach it.

Now consider values of x slightly less (i.e., "to the left of") 3. There, the value of f is a negative number with large magnitude. (For example, $f(2.999) = 1.999/-0.001 = -1999.$) As x decreases from 3, f's value remains negative but becomes smaller in magnitude (e.g., $f(2.5) = -30$, $f(2) = -1$, $f(1.5) = -1/3$) until it hits the x-axis at $x = 1$ (which, after all, is f's zero). As x keeps decreasing "to the left of" 1, $f(x)$ becomes positive and its magnitude starts increasing: $f(0.5) = 1/5$, $f(0) = 1/3$, $f(-1) = 1/2$, $f(-5) = 3/4$, $f(-100) \approx 0.998$, etc. Indeed, as $x < 3$ decreases, $f(x)$ will approach 1 from "below", but never quite reach it.

From this discussion, it appears that the range of f includes all real numbers except 1. However, we must also take into account the fact that f has a "hole" at $x = -2$. What is the y-coordinate of that hole? We can find out by plugging in -2 for x in the simplified version of f (that we obtained by cancelling the $x + 2$ factor from numerator and denominator). Doing so, we get $(-2-1)/(-2-3) = -3/-5 = 3/5$. Unless there is some value of x other than -2 such that $(x-1)/(x-3) = 3/5$, that would mean that $3/5$ (in addition to 1) is not in the range of f. Our graph of f, based upon the discussion above (see below), passes the horizontal line test, which implies that, if the graph is accurate in that respect, f is **one-to-one** and therefore -2 is the unique value of x that produces $3/5$. Which is to say that the range of f is $\mathcal{R} - \{1, 3/5\}$.

To verify that f is actually one-to-one, one can show that, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. By employing the fact that $u/v = y/z$ implies $uz = vy$, this is not difficult and is left as an exercise.

Notice that the graph of f here, generated by a graphing device, does not explicitly show the hole at $(-2, 3/5)$.



9. Match the equation with the graph.
(a) $y = \frac{1}{2}(x-1)^3(x+2)$
(b) $y = -(x-1)^2(x+2)$
(c) $y = (x-1)(x+2)$

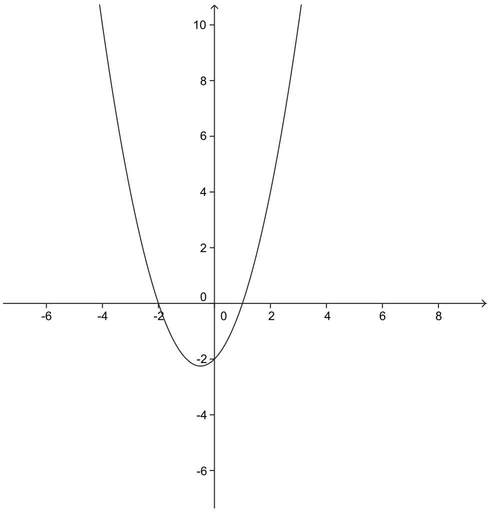


Figure 1:

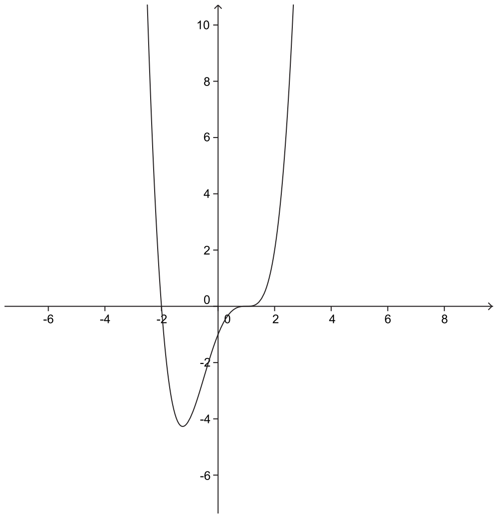


Figure 2:

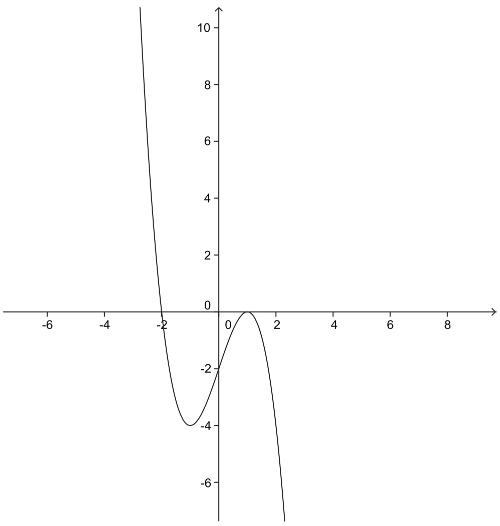


Figure 3:

Solution: All three graphs are of functions having zeros at $x = -2$ and $x = 1$, and nowhere else. So, on that count alone, there is nothing to distinguish between them.

However, in Figure 3 the zero at $x = 1$ appears to be of even multiplicity, as the graph touches the x-axis but does not pass through it. This is consistent with **(b)** alone. Further evidence of this match is that the end behavior of the graph in Figure 3 is consistent with a degree three polynomial having a negative leading coefficient, such as **(b)** describes.

In Figure 2, the zero at $x = 1$ appears to correspond to a zero of odd multiplicity greater than two, as the graph flattens out as it passes through the x-axis. This is consistent with **(a)** alone. Further evidence of this match is that the graph's end behavior is consistent with an even-degree polynomial with positive leading coefficient, such as **(a)** describes.

Finally, the zero at $x = 1$ in the graph of Figure 1 appears to be simple, which is consistent with **(c)** alone. Further evidence of this match is that the graph's end behavior is consistent with an even-degree polynomial with positive leading coefficient, such as **(c)** describes.

So the answer is that **(a)** matches Figure 2, **(b)** matches Figure 3, and **(c)** matches Figure 1.