MATH 114 Calculus
Theorems and Laws from Chapter 2 of Stewart

Limit Laws: If c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

1. $\lim_{x\to a}[f(x)+g(x)] \;=\; \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] \;=\; \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a} c\cdot f(x) \;=\; c\cdot \lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)\cdot g(x)] \;=\; \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a}[f(x)/g(x)] \;=\; \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a}[f(x)]^n \;=\; [\lim_{x\to a} f(x)]^n$ for a positive integer n

7. $\lim_{x\to a} c \;=\; c$

8. $\lim_{x\to a} x \;=\; a$

9. $\lim_{x\to a} x^n \;=\; a^n$ for a positive integer n

10. $\lim_{x\to a} \sqrt[n]{x} \;=\; \sqrt[n]{a}$

11. $\lim_{x\to a} \sqrt[n]{f(x)} \;=\; \sqrt[n]{\lim_{x\to a} f(x)}$

Theorem 2.3.0: If, on some interval (b,c) that includes a, $f(x)=g(x)$ except possibly at $x=a$, then

$$\lim_{x\to a} f(x) \;=\; \lim_{x\to a} g(x)$$

(assuming that either limit exists).

In part, what this theorem says is that $\lim_{x\to a} f(x)$ does not depend upon $f(a)$ (or whether it is even defined).

Theorem 2.3.1:

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L$$

In other words, the (two-sided) limit exists if and only if both of the one-sided limits exist and agree with each other.

Theorem 2.3.2: Let (b,c) be an interval including a in which, for all x (except possibly a), $f(x) \le g(x)$. Then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

assuming that both limits exist.

Theorem 2.3.3 (Squeeze/Sandwich/Pinching): Let (b, c) be an interval including a in which, for all x (except possibly a), $f(x) \le g(x) \le h(x)$. Also assume that

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

Then

$$\lim_{x\to a} g(x) = L$$

Definition 2.4.2 (Limit): Let f be a function defined on some open interval that contains a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) \ = \ L$$

means that, for every number $\epsilon > 0$ (no matter how small) there exists a number $\delta > 0$ such that if x lies in the interval $(a - \delta, a + \delta)$ (but excluding a itself) then $f(x)$ lies in the interval $(L - \epsilon, L + \epsilon)$.

Definition 2.4.3 (Left-hand Limit): Let f be a function defined on some open interval that contains a, except possibly at a itself. Then

$$\lim_{x\to a^-} f(x) \ = \ L$$

means that, for every number $\epsilon > 0$ (no matter how small) there exists a number $\delta > 0$ such that if x lies in the interval $(a - \delta, a)$ then $f(x)$ lies in the interval $(L - \epsilon, L + \epsilon)$.

Definition 2.4.4 (Right-hand Limit): Let f be a function defined on some open interval that contains a, except possibly at a itself. Then

$$\lim_{x\to a^+} f(x) \ = \ L$$

means that, for every number $\epsilon > 0$ (no matter how small) there exists a number $\delta > 0$ such that if x lies in the interval $(a, a + \delta)$ then $f(x)$ lies in the interval $(L - \epsilon, L + \epsilon)$.

Definition 2.4.6 (Infinite Limit): Let f be defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = +\infty$$

means that for every positive number M there exists a positive number δ such that if x lies in the interval $(a - \delta, a + \delta)$, then $f(x) > M$.

Definition 2.4.7 (Infinite Limit (negative)): Let f be defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

means that for every negative number N there exists a positive number δ such that if x lies in the interval $(a - \delta, a + \delta)$, then $f(x) < N$.

Definition 2.5.1 (Continuity at a point): A function f is continuous at a if

$$\lim_{x\to a} f(x) = f(a)$$

Definition 2.5.3: f is **continuous on an interval** if it is continuous at every number in the interval.

Theorem 2.5.4 (preservation of continuity by arithmetic operations): If f and g are continuous at a, then so are the functions $f+g$, $f-g$, cf (where c is a constant), fg, and f/g (if $g(a) \neq 0$)

Theorem 2.5.5 Each polynomial is continuous everywhere; each rational function (i.e., one of the form p/q where p and q are polynomials) is continuous on its domain.

Theorem 2.5.7 Each function of any of the following types is continuous on its domain: polynomial, rational function, root function (i.e., of form $\sqrt[n]{x}$), and trigonometric (e.g., $\sin x$, $\cos x$, etc.)

Theorem 2.5.8 If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) \;=\; f(\lim_{x\to a} g(x))$$

Theorem 2.5.9 If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. In other words, if

$$\lim_{x\to a} g(x) = g(a) \quad \text{and} \quad \lim_{x\to g(a)} f(x) = f(g(a))$$

then

$$\lim_{x\to a} (f \circ g)(x) = (f \circ g)(a) = f(g(a))$$

Theorem 2.5.10 (The Intermediate Value Theorem (IMVT)): Let f be continuous on the closed interval $[a, b]$, where $f(a) \neq f(b)$ and let N be any number between $f(a)$ and $f(b)$. (That is, either $f(a) < N < f(b)$ or $f(a) > N > f(b)$.) Then there exists $c \in (a, b)$ such that $f(c) = N$.