Part A: Predicate Strength.

Recall that a *predicate* is simply a function that yields a boolean value and that $\lbrack \cdot \rbrack$ is the *everywhere* operator on predicates; i.e., for a predicate $P$, the expression $\lbrack P \rbrack$ is true if $P$ holds in all states (i.e., everywhere) but false if there is at least one state in which $P$ does not hold.

Let $P$ and $Q$ be predicates. Then, with respect to weakness/strength, the possible relationships between them are as follows:

If $\lbrack P \Rightarrow Q \rbrack$ (equivalently, $\lbrack Q \Leftarrow P \rbrack$), we say that $P$ is *stronger than* $Q$ (equivalently, $Q$ is *weaker than* $P$).

If each of $P$ and $Q$ is stronger than the other, we say that they are *equivalent*. This makes sense, because

$$\lbrack P \Rightarrow Q \rbrack \land \lbrack P \Leftarrow Q \rbrack \equiv \lbrack P \equiv Q \rbrack$$

If, on the other hand, $P$ is stronger than $Q$ but it is not the case that $Q$ is stronger than $P$, we say that $P$ is *strictly stronger than* $Q$. Equivalently, if $Q$ is weaker than $P$ but it is not the case that $P$ is weaker than $Q$, we say that $Q$ is *strictly weaker than* $P$.

If neither $P$ is stronger than $Q$ nor $Q$ is stronger than $P$, then $P$ and $Q$ are unrelated with respect to weakness/strength.

These are summarized in the following table:

<table>
<thead>
<tr>
<th>$\lbrack P \Rightarrow Q \rbrack$</th>
<th>$\lbrack P \Leftarrow Q \rbrack$</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>true</td>
<td>true</td>
<td>$P$ and $Q$ are equivalent</td>
</tr>
<tr>
<td>false</td>
<td>true</td>
<td>$P$ is strictly weaker than $Q$</td>
</tr>
<tr>
<td>true</td>
<td>false</td>
<td>$P$ is strictly stronger than $Q$</td>
</tr>
<tr>
<td>false</td>
<td>false</td>
<td>$P$ and $Q$ are unrelated</td>
</tr>
</tbody>
</table>

In order to demonstrate that, for example, $\lbrack P \Rightarrow Q \rbrack$ is false, it suffices to identify a state that satisfies $P$ but fails to satisfy $Q$.

In order to demonstrate that, for example, $\lbrack P \Rightarrow Q \rbrack$ is true, it suffices to show that $Q$ is satisfied in every state satisfying $P$, or, equivalently, that there exists no state in which $P$ is satisfied but $Q$ is not satisfied.

For each given pair $P$ and $Q$ of predicates below, indicate the weakness/strength relationship that exists between them. Justify your answers.

(1) $P: x \geq 0$ and $Q: x \geq 1$
(2) $P: x \geq 0 \land y < x$ and $Q: x \geq 1$
(3) $P: x \geq 1 \land y < x$ and $Q: x \geq 0$
(4) $P : x \geq 0$ and $Q : x^2 + y^2 > 0$
(5) $P : x \geq 0$ and $Q : x^2 + y^2 \geq 0$
(6) $P : x \geq 1 \Rightarrow x \geq 0$ and $Q : x \geq 1$
(7) $P : x \geq 1$ and $Q : (\exists i \mid i \geq 1 : x = i)$
(8) $P : (\forall i \mid R : Q)$ and $(\exists i \mid R : Q)$, where $R$ is not equivalent to $false$.

Part B: Developing Predicates

For each of the following (informally stated) predicates (some of which are actually propositions in that they depend on no arguments), express it formally, using the language of predicate logic.

In writing your answers, you must supply a definition for any function/operator you use that has not been defined already, either by you or by virtue of it being “pre-defined”. (Of course, any such definition should be written in terms of previously-defined concepts!) You may consider the following to be pre-defined:

1. the logical connectives $\equiv$ (equivalence), $\land$ (conjunction), $\lor$ (disjunction), $\Rightarrow$ (implication), $\Leftarrow$ (consequence), $\neg$ (negation)
2. the arithmetic operators $+$ (addition), $-$ (subtraction), $\cdot$ (multiplication), $\div$ (division), $\min$ (minimum), $\max$ (maximum)
3. the relational operators $=$ (equals), $<$ (less than), $>$ (greater than)
4. quantification over any operator that is both symmetric (i.e., commutative) and associative, including $+, \cdot, \min, \max, \land$ (which is usually written $\forall$, and $\lor$ (which is usually written $\exists$).
5. the “number of” quantifier, as in $(\#x \mid R : P)$

For purposes of brevity, you can write, for example, $a \leq b$ rather than $a < b \lor a = b$ and $a \neq b$ rather than $\neg(a = b)$.

As an example, if you were to express formally the predicate $p$ is the largest prime number, one good answer would be

$$isLargestPrime.p : isPrime.p \land \neg(\exists r \mid isPrime.r : r > p)$$

where $isPrime.m : \neg(\exists k \mid 1 < k < m : k|m)$, where $a|b : (\exists c \mid a \cdot c = b)$.

End of sample answer.

As an aside, note that $isLargestPrime$ is false everywhere, as there is no largest prime number.

Problems:

1. There exists a smallest natural number.
2. (Goldbach’s Conjecture) Every even number greater than two is the sum of two prime numbers.
3. $p$ is the $n$-th largest prime (i.e., there are exactly $n - 1$ prime numbers smaller than $p$).
4. For every pair of (distinct) real numbers, there is another real number falling between them.
5. The first occurrence of $x$ in array segment $f[p..q]$ is at location $m$.
6. In the integer array $f$, the sum of the elements in segment $f[p..q]$ is at least as large as that of any segment of $f$.
7. Every value occurring in array $f$ also occurs in array $g$.
8. The number of distinct values occurring in $f[p..q]$ is $m$. 
