SE 504 (Formal Methods and Models)
Spring 2009
HW #1
Due: Thursday, Feb. 12 (7:20pm)

For each of the following (informally stated) predicates (a few of which are actually propositions in that they depend on no arguments), express it formally, using the language of predicate logic.

You must supply a definition for any function/predicate/operator that is used in your answers, unless it is considered to be "pre-defined". In addition to the *logical connectives* ($\equiv$ (equivales), $\wedge$ (conjunction), $\vee$ (disjunction), $\Rightarrow$ (implication), $\Leftarrow$ (consequence), and $\neg$ (negation)) and the *logical quantifiers* ($\forall$ (universal) and $\exists$ (existential)), you may consider the following to be pre-defined:

1. the arithmetic operators $+$ (addition), $-$ (subtraction), $\cdot$ (multiplication), $/$ (or $\div$) (division), `min` (minimum), `max` (maximum)
2. the relational operators $=$ (equals), $<$ (less than), $>$ (greater than)
3. *quantification* over any operator that is both symmetric (i.e., commutative) and associative, including $+$, $\cdot$, `min`, `max`.
4. the "counting" quantifier, as in $(\#x \mid R : P)$, which is shorthand for $(+x \mid R \wedge P \ : \ 1)$.

For purposes of brevity, you may write, for example, $a \leq b$ rather than $a < b \ \vee \ a = b$ and $a \neq b$ rather than $\neg(a = b)$.

As an example, if you were to express formally the predicate *p is the largest prime number*, one good answer would be

$$isLargestPrime.p \ \widehat{=} \ isPrime.p \ \wedge \ \neg(\exists r \mid isPrime.r \ : \ r > p)$$

where $isPrime.m \ \widehat{=} \ m > 1 \ \wedge \ \neg(\exists k \mid 1 < k < m \ : \ k|m)$,
where $a|b \ \widehat{=} \ (\exists c \mid : \ a \cdot c = b)$.

The assumption here is that all variables (including dummies) are of type $\mathcal{Z}$ (or `int`, if you prefer). Note that the symbol $\widehat{=}$ is intended to be read as "is defined by". Incidentally, *isLargestPrime* is false everywhere (i.e., for any supplied value of p), as there is no largest prime number.
End of sample answer.

Problems:

1. There exists a smallest integer.
2. Every prime number can be written as the sum of two composite (i.e., non-prime) numbers.
3. k can be written as the product of three prime numbers.
4. $\oplus$ is not associative, where $\oplus$ is some logical connective (like $\wedge$, $\vee$, $\Rightarrow$, etc.). (Exactly what function $\oplus$ denotes is irrelevant.)
5. In the array segment $f[m..n)$ (which is shorthand for $f[m..n-1]$) there are at least four occurrences of values greater than x.

6. The next-to-last occurrence of x in array segment $f[p..q)$ is at location m.
7. For every pair of (distinct) real numbers, there is another real number falling exactly halfway between them.
8. The number of distinct values occurring in array f is m.
9. Every value occurring in array f also occurs in array g.

We denote the length of an array f by $\#f$ (or by $f.length$, if you prefer); hence, the index range of f begins at zero and ends at $\#f - 1$.