

## SE 504 (Formal Methods and Models)

### The strength/weakness relationship between predicates

Recall that a *predicate* is simply a function that yields a boolean value and that  $[\cdot]$  is the *everywhere* operator on predicates; i.e., for a predicate  $P$ , the expression  $[P]$  is true if  $P$  holds in all states (i.e., everywhere) but false if there is at least one state in which  $P$  does not hold. Technically,

$$[P] \hat{=} (\forall x_1 : t_1, x_2 : t_2, \dots, x_n : t_n \mid : P)$$

where the  $x_i$ 's are precisely those variables having at least one free occurrence in  $P$  and the  $t_i$ 's are their respective types.

Let  $P$  and  $Q$  be predicates. Then, with respect to weakness/strength, the possible relationships between them are as follows:

If  $[P \Rightarrow Q]$  (equivalently,  $[Q \Leftarrow P]$ ), we say that  $P$  is *stronger than*  $Q$  (equivalently,  $Q$  is *weaker than*  $P$ ).

If each of  $P$  and  $Q$  is stronger than the other, we say that they are *equivalent*. This makes sense, because

$$[P \Rightarrow Q] \wedge [P \Leftarrow Q] \equiv [P \equiv Q]$$

If, on the other hand,  $P$  is stronger than  $Q$  but it is not the case that  $Q$  is stronger than  $P$ , we say that  $P$  is *strictly stronger than*  $Q$ . Equivalently, if  $Q$  is weaker than  $P$  but it is not the case that  $P$  is weaker than  $Q$ , we say that  $Q$  is *strictly weaker than*  $P$ .

If neither  $P$  is stronger than  $Q$  nor  $Q$  is stronger than  $P$ , then  $P$  and  $Q$  are unrelated with respect to weakness/strength.

These are summarized in the following table:

$[P \Rightarrow Q]$	$[P \Leftarrow Q]$	Relationship
true	true	$P$ and $Q$ are equivalent
false	true	$P$ is strictly weaker than $Q$
true	false	$P$ is strictly stronger than $Q$
false	false	$P$ and $Q$ are unrelated

In order to demonstrate that, for example,  $[P \Rightarrow Q]$  is false, it suffices to identify a state in which  $P \Rightarrow Q$  is false (i.e., a state that satisfies  $P$  but fails to satisfy  $Q$ ).

In order to demonstrate, for example,  $[P \Rightarrow Q]$ , it suffices to prove  $P \Rightarrow Q$ . See Metatheorem 9.16 (and the accompanying discussion) in the text by Gries and Schneider. Such a proof can be of the form taught in the aforementioned text, or it could be a little less formal. One could, for example, consider an arbitrary state  $s$  satisfying  $P$  and argue persuasively that  $s$  also satisfies  $Q$ . (**Note:** An “arbitrary” state is one about which nothing can be assumed; students often make the mistake of choosing a particular state (having properties convenient to their purposes) and calling it “arbitrary”).

As an example, suppose we have  $P : x \geq 1 \wedge y < x$  and  $Q : x \geq 0$ , with  $x$  and  $y$  of type **integer**. Then  $P$  is strictly stronger than  $Q$ , because  $[P \Rightarrow Q]$  holds but  $[P \Leftarrow Q]$  does not. The latter follows from  $P \Leftarrow Q$  being false in any state in which  $x = 0$ . The former is rather obvious, but may be proved rigorously as follows:

$$\begin{aligned}
 & x \geq 0 \\
 = & \quad \langle \text{integer arithmetic} \rangle \\
 & x = 0 \vee x \geq 1 \\
 \Leftarrow & \quad \langle p \Rightarrow p \vee q \text{ (Gries 3.76a)} \rangle \\
 & x \geq 1 \\
 \Leftarrow & \quad \langle p \wedge q \Rightarrow p \text{ (Gries 3.76b)} \rangle \\
 & x \geq 1 \wedge y < x
 \end{aligned}$$