Chapter Ten:
Grammars

Grammar is another of those common words for which the study of formal language introduces a precise technical definition. For us, a grammar is a certain kind of collection of rules for building strings. Like DFAs, NFAs, and regular expressions, grammars are mechanisms for defining languages rigorously.

A simple restriction on the form of these grammars yields the special class of right-linear grammars. The languages that can be defined by right-linear grammars are exactly the regular languages. There it is again!

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- 10.1 A Grammar Example for English
- 10.2 The 4-Tuple
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- 10.6 Every Right-linear Grammar Generates a Regular Language

A Little English

- An article can be the word *a* or *the*:

  $A \rightarrow a$
  $A \rightarrow the$

- A noun can be the word *dog*, *cat* or *rat*:

  $N \rightarrow dog$
  $N \rightarrow cat$
  $N \rightarrow rat$

  A noun phrase is an article followed by a noun:

  $P \rightarrow AN$

A Little English

- An verb can be the word *loves, hates* or *eats*:

  $V \rightarrow loves$
  $V \rightarrow hates$
  $V \rightarrow eats$

  A sentence can be a noun phrase, followed by a verb, followed by another noun phrase:

  $S \rightarrow PVP$

The Little English Grammar

- Taken all together, a grammar G_1 for a small subset of unpunctuated English:

$S \rightarrow PVP$	$A \rightarrow a$
$P \rightarrow AN$	$A \rightarrow the$
$V \rightarrow loves$	$N \rightarrow dog$
$V \rightarrow hates$	$N \rightarrow cat$
$V \rightarrow eats$	$N \rightarrow rat$

- Each *production* says how to modify strings by substitution
- $x \rightarrow y$ says, substring x may be replaced by y

$$
\begin{array}{ll}
S \rightarrow PVP & A \rightarrow a \\
P \rightarrow AN & A \rightarrow the \\
V \rightarrow loves & N \rightarrow dog \\
V \rightarrow hates & N \rightarrow cat \\
V \rightarrow eats & N \rightarrow rat
\end{array}
$$

- Start from S and follow the productions of G_1
- This can derive a variety of (unpunctuated) English sentences:

$$S \Rightarrow PVP \Rightarrow ANVP \Rightarrow theNVP \Rightarrow thecatVP \Rightarrow thecateatsP \Rightarrow thecateatsAN \Rightarrow thecateatsaN \Rightarrow thecateatsarat$$

$$S \Rightarrow PVP \Rightarrow ANVP \Rightarrow aNVP \Rightarrow adogVP \Rightarrow adoglovesP \Rightarrow adoglovesAN \Rightarrow adoglovestheN \Rightarrow adoglovesthecat$$

$$S \Rightarrow PVP \Rightarrow ANVP \Rightarrow theNVP \Rightarrow thecatVP \Rightarrow thecathatesP \Rightarrow thecathatesAN \Rightarrow thecathatestheN \Rightarrow thecathatesthedog$$

$S \rightarrow PVP$	$A \rightarrow a$
$P \rightarrow AN$	$A \rightarrow the$
$V \rightarrow loves$	$N \rightarrow dog$
$V \rightarrow hates$	$N \rightarrow cat$
$V \rightarrow eats$	$N \rightarrow rat$

- Often there is more than one place in a string where a production could be applied
- For example, *PlovesP*:
  - $PlovesP \Rightarrow ANlovesP$
  - $PlovesP \Rightarrow PlovesAN$
- The derivations on the previous slide chose the leftmost substitution at every step, but that is not a requirement
- The language defined by a grammar is the set of lowercase strings that have at least one derivation from the start symbol *S*

$$
\begin{array}{l}
S \rightarrow PVP \\
P \rightarrow AN \\
V \rightarrow loves \mid hates \mid eats \\
A \rightarrow a \mid the \\
N \rightarrow dog \mid cat \mid rat
\end{array}
$$

- Often, a grammar contains more than one production with the same left-hand side
- Those productions can be written in a compressed form
- The grammar is not changed by this
- This example still has ten productions

Informal Definition

A *grammar* is a set of productions of the form $x \rightarrow y$. The strings x and y can contain both lowercase and uppercase letters; x cannot be empty, but y can be ε. One uppercase letter is designated as the start symbol (conventionally, it is the letter S).

- Productions define permissible string substitutions
- When a sequence of permissible substitutions starting from S ends in a string that is all lowercase, we say the grammar generates that string
- $L(G)$ is the set of all strings generated by grammar G

$$
\begin{aligned}
&S \rightarrow aS \\
&S \rightarrow X \\
&X \rightarrow bX \\
&X \rightarrow \\
&\varepsilon
\end{aligned}
$$

- That final production for X says that X may be replaced by the empty string, so that for example $abbX \Rightarrow abb$
- Written in the more compact way, this grammar is:

$$
\begin{aligned}
&S \rightarrow aS \mid X \\
&X \rightarrow bX \mid \\
&\varepsilon
\end{aligned}
$$

$$S \to aS \mid X$$
$$X \to bX \mid \varepsilon$$

$$S \Rightarrow aS \Rightarrow aX \Rightarrow a$$

$$S \Rightarrow X \Rightarrow bX \Rightarrow b$$

$$S \Rightarrow aS \Rightarrow aX \Rightarrow abX \Rightarrow abbX \Rightarrow abb$$

$$S \Rightarrow aS \Rightarrow aaS \Rightarrow aaaS \Rightarrow aaaX \Rightarrow$$
$$aaabX \Rightarrow aaabbX \Rightarrow aaabb$$

$$
\begin{array}{l}
S \rightarrow aS \mid X \\
X \rightarrow bX \mid \\
\varepsilon
\end{array}
$$

- For this grammar, all derivations of lowercase strings follow this simple pattern:
  - First use $S \rightarrow aS$ zero or more times
  - Then use $S \rightarrow X$ once
  - Then use $X \rightarrow bX$ zero or more times
  - Then use $X \rightarrow \varepsilon$ once
- So the generated string always consists of zero or more *a*s followed by zero or more *b*s
- $L(G) = L(a^*b^*)$

Untapped Power

- All our examples have used productions with a single uppercase letter on the left-hand side
- Grammars can have any non-empty string on the left-hand side
- The mechanism of substitution is the same
  - $Sb \rightarrow bS$ says that bS can be substituted for Sb
- Such productions can be very powerful, but we won't need that power yet
- We'll concentrate on grammars with one uppercase letter on the left-hand side of every production

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Formalizing Grammars

- Our informal definition relied on the difference between lowercase and uppercase
- The formal definition will use two separate alphabets:
  - The *terminal symbols (*typically lowercase)
  - The *nonterminal symbols* (typically uppercase)
- So a formal grammar has four parts…

4-Tuple Definition

- A *grammar* G is a 4-tuple $G = (V, \Sigma, S, P)$, where:
  - V is an alphabet, the *nonterminal alphabet*
  - Σ is another alphabet, the *terminal alphabet*, disjoint from V
  - $S \in V$ is the *start symbol*
  - P is a finite set of productions, each of the form $x \rightarrow y$, where x and y are strings over $\Sigma \cup V$ and $x \neq \varepsilon$

Example

$$S \to aS \mid X$$
$$X \to bX \mid$$
$$\varepsilon$$

- Formally, this is $G = (V, \Sigma, S, P)$, where:
  - $V = \{S, X\}$
  - $\Sigma = \{a, b\}$
  - $P = \{S \to aS, S \to X, X \to bX, X \to \varepsilon\}$
- The order of the 4-tuple is what counts:
  - $G = (\{S, X\}, \{a, b\}, S, \{S \to aS, S \to X, X \to bX, X \to \varepsilon\})$

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The Program

- For DFAs, we derived a zero-or-more-step δ^* function from the one-step δ
- For NFAs, we derived a one-step relation on IDs, then extended it to a zero-or-more-step relation
- We'll do the same kind of thing for grammars…

$w \Rightarrow z$

- Defined for a grammar $G = (V, \Sigma, S, P)$
- $\Rightarrow$ is a relation on strings
- $w \Rightarrow$ z ("*w derives z*") if and only if there exist strings u, x, y, and v over $\Sigma \cup V$, with
  - $w = uxv$
  - $z = uyv$
  - $(x \rightarrow \text{y}) \in P$
- That is, *w* can be transformed into *z* using one of the substitutions permitted by *G*

Derivations And $w \Rightarrow^* z$

- A sequence of $\Rightarrow$-related strings $x_0 \Rightarrow x_1 \Rightarrow \ldots \Rightarrow x_n$, is an n-step *derivation*
- $w \Rightarrow^* z$ if and only if there is a derivation of 0 or more steps that starts with w and ends with z
- That is, w can be transformed into z using a sequence of zero or more of the substitutions permitted by G

$L(G)$

- The language generated by a grammar G is $L(G) = \{x \in \Sigma^* \mid S \Rightarrow^* x\}$
- That is, the set of fully terminal strings derivable from the start symbol
- Notice the restriction $x \in \Sigma^*$:
  - The intermediate strings in a derivation can use both Σ and V
  - But only the fully terminal strings are in $L(G)$

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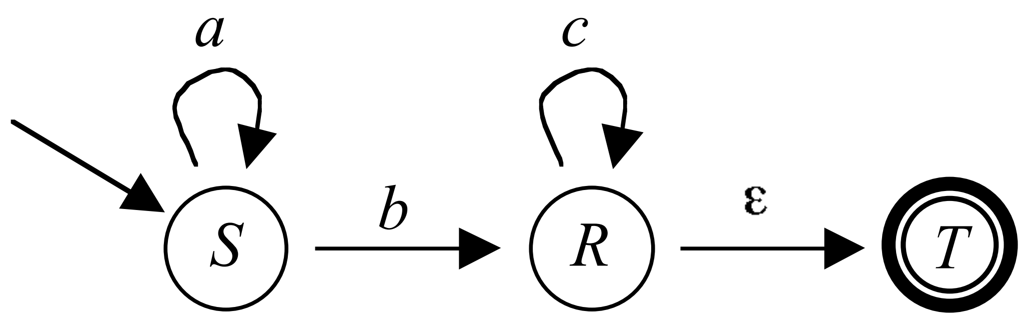
NFA to Grammar

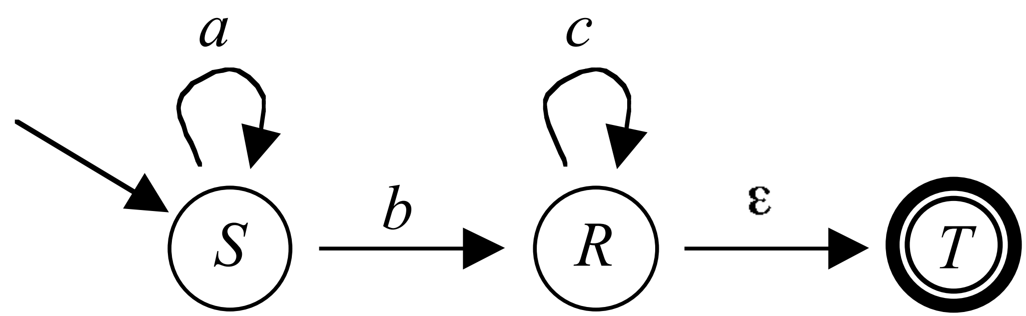
- To show that there is a grammar for every regular language, we will show how to convert any NFA into an equivalent grammar
- That is, given an NFA M, construct a grammar G with $L(M) = L(G)$
- First, an example…

Example:



- The grammar we will construct generates *L*(*M*)
- In fact, its derivations will mimic what *M* does
- For each state, our grammar will have a nonterminal symbol (*S*, *R* and *T*)
- The start state will be the grammar's start symbol
- The grammar will have one production for each transition of the NFA, and one for each accepting state

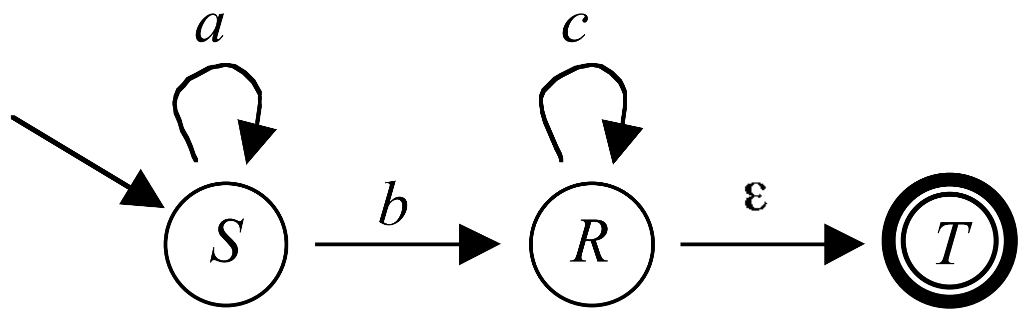
Example:



- For each possible transition $Y \in \delta(X,z)$ in the NFA, our grammar has a production $X \rightarrow zY$
- That gives us these four to start with:

Transition of M	Production in G
$\delta(S,a) = \{S\}$	$S \rightarrow aS$
$\delta(S,b) = \{R\}$	$S \rightarrow bR$
$\delta(R,c) = \{R\}$	$R \rightarrow cR$
$\delta(R,\varepsilon) = \{T\}$	$R \rightarrow T$

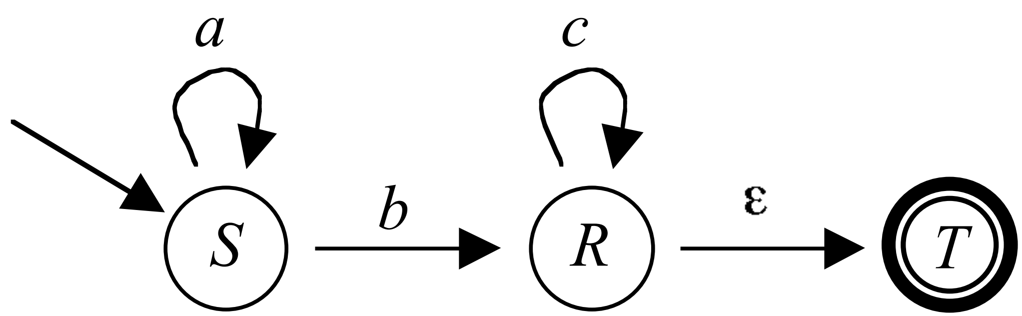
Example:



- In addition, for each accepting state in the NFA, our grammar has an ε-production
- That adds one more:

Accepting state of *M*	Production in *G*
T	$T \to \varepsilon$

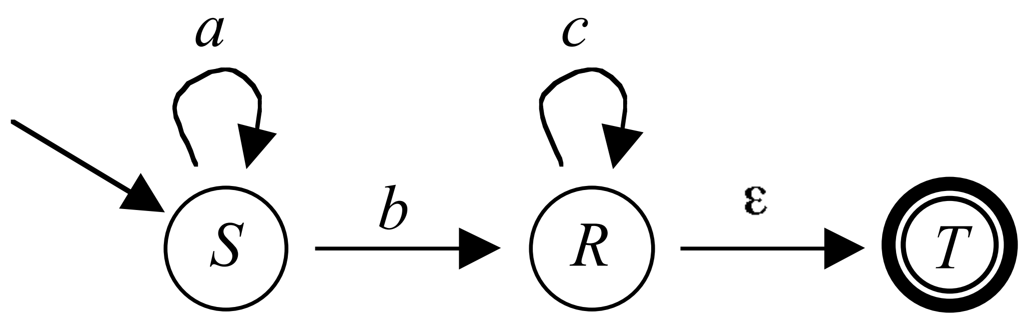
Example:



- The complete grammar has one production for each transition, and one for each accepting state:

$$
\boxed{\begin{array}{l}
S \rightarrow aS \\
S \rightarrow bR \\
R \rightarrow cR \\
R \rightarrow T \\
T \rightarrow \varepsilon
\end{array}}
$$

$S \rightarrow aS$
$S \rightarrow bR$
$R \rightarrow cR$
$R \rightarrow T$
$T \rightarrow \varepsilon$



- Compare the behavior of *M* as it accepts *abc* with the behavior of *G* as it generates *abc*:

$$(S,abc) \mapsto (S,bc) \mapsto (R,c) \mapsto (R,\varepsilon) \mapsto (T,\varepsilon)$$
$$S \Rightarrow aS \Rightarrow abR \Rightarrow abcR \Rightarrow abcT \Rightarrow abc$$

- Every time the NFA reads a symbol, the grammar generates that symbol
- In general, *M* can be in state *Y* after reading string *x* if and only if *G* can derive the string *xY*

Theorem 10.4

Every regular language is generated by some grammar.

- Proof is by construction; let $M = (Q, \Sigma, \delta, S, F)$ be any NFA
- Construct $G = (Q, \Sigma, S, P)$
  - Q, Σ, and S are the same as for M
  - P is constructed from δ and F:
    - Wherever M has $Y \in \delta(X,z)$, P contains $X \to zY$
    - And for each $X \in F$, P contains $X \to \varepsilon$
- Now G has $X \to zY$ whenever $(X,z) \mapsto (Y,\varepsilon)$
- By induction we can extend this to any string $z \in \Sigma^*$:
  G has $X \to^* zY$ whenever $(X,z) \mapsto^* (Y,\varepsilon)$
- And by construction, G has $Y \to \varepsilon$ whenever M has $Y \in F$
- So for all strings $z \in \Sigma^*$, $\delta^*(S,z)$ contains at least one element of F if and only if $S \Rightarrow^* z$
- $L(M) = L(G)$

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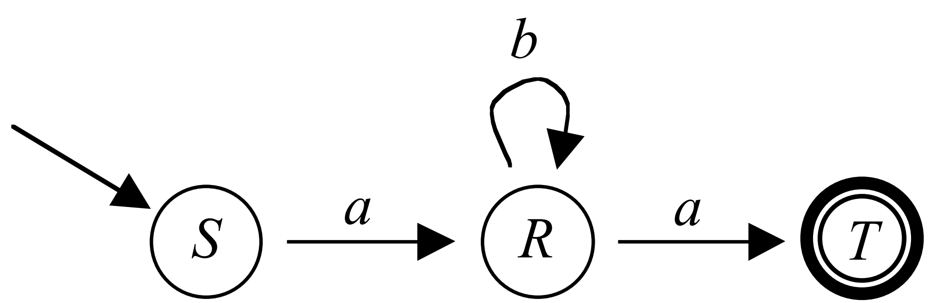
Single-Step Grammars

- A grammar $G = (V, \Sigma, S, P)$ is *single step* if and only if every production in P is in one of these three forms, where $X \in V$, $Y \in V$, and $z \in \Sigma$:
  - $X \rightarrow zY$
  - $X \rightarrow z$
  - $X \rightarrow \varepsilon$
- Given any single-step grammar, we could run the previous construction backwards, building an equivalent NFA…

Reverse Example

$S \rightarrow aR$
$R \rightarrow bR$
$R \rightarrow aT$
$T \rightarrow \varepsilon$

- This grammar generates $L(ab^*a)$:
- All its productions are of the kinds built in our construction
- Running the construction backwards, we get three states S, R, and T
- The first three productions give us the three arrows, and the fourth makes T accepting:



Production Massage

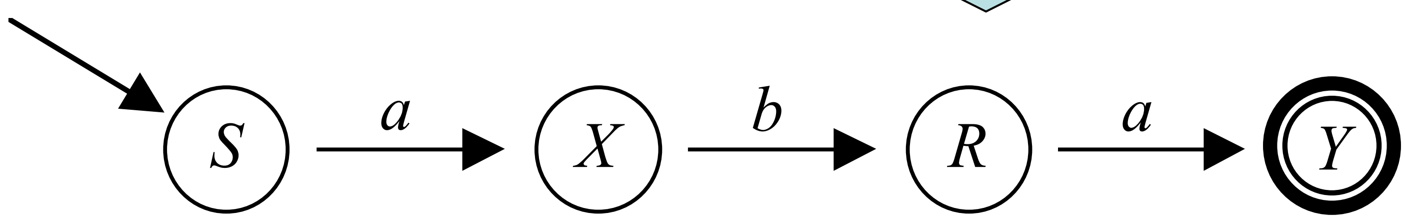
$S \rightarrow abR$
$R \rightarrow a$

- Even if all the productions are not of the required form, it is sometimes possible to massage them until they are
- $S \rightarrow abR$ does not have the right form:
  - Equivalent productions $S \rightarrow aX$ and $X \rightarrow bR$ do
- $R \rightarrow a$ does not have the right form:
  - Equivalent productions $R \rightarrow aY$ and $Y \rightarrow \varepsilon$ do
- After those changes we can run the construction backwards…

Massaged Reverse Example

$S \rightarrow abR$
$R \rightarrow a$

$S \rightarrow aX$
$X \rightarrow bR$
$R \rightarrow aY$
$Y \rightarrow \varepsilon$



Right-Linear Grammars

- A grammar $G = (V, \Sigma, S, P)$ is *right linear* if and only if every production in P is in one of these two forms, where $X \in V$, $Y \in V$, and $z \in \Sigma^*$:
  - $X \to zY$, or
  - $X \to z$
- So every production has:
  - A single nonterminal on the left
  - At most one nonterminal on the right, and only as the rightmost symbol
- Note that this includes all single-step grammars
- This special form makes it easy to massage the productions and then transform them into NFAs

Lemma 10.5

Every right-linear grammar G is equivalent to some single-step grammar G'.

- Proof is by construction
- Let $G = (V, \Sigma, S, P)$ be any right-linear grammar
- Each production is $X \to z_1 \ldots z_n \omega$, where $\omega \in V$ or $\omega = \varepsilon$
- For each such production, let P contains these n+1 productions, where each K_i is a new nonterminal symbol:
- Now let $G = (V', \Sigma, S, P')$, where V' is the set of nonterminals used in P'
- Any step of a derivation G is equivalent to the corresponding n+1 steps in G'
- The reverse is true for derivations of terminal strings in G'
- So $L(G) = L(G')$

$$
\begin{array}{l}
X \to z_1 K_1 \\
K_1 \to z_2 K_2 \\
\ldots \\
K_{n-1} \to z_n K_n \\
K_n \to \omega
\end{array}
$$

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Theorem 10.6

For every right-linear grammar G, $L(G)$ is regular.

- Proof is by construction
- Use Lemma 10.5 to get single-step form, then use the reverse of the construction from Theorem 10.4

Left-Linear Grammars

- A grammar $G = (V, \Sigma, S, P)$ is *left linear* if and only if every production in P is in one of these two forms, where $X \in V$, $Y \in V$, and $z \in \Sigma^*$:
  - $X \rightarrow Yz$, or
  - $X \rightarrow z$
- This parallels the definition of right-linear
- With a little more work, one can show that the language generated is also always regular

Regular Grammars, Regular Languages

- Grammars that are either left-linear or right-linear are called *regular grammars*
- A simple inspection tells you whether G is a regular grammar; if it is, $L(G)$ is a regular language
- Note that if G is not a regular grammar, that tells you nothing: $L(G)$ might still be regular language
- This example is not right-linear and not left-linear, but $L(G)$ is the regular language $L((aaa)^*)$:

$$\boxed{S \rightarrow aSaa \mid \varepsilon}$$

The Next Big Question

- We know that all regular grammars generate regular languages
- We've seen a non-regular grammar that still generates a regular language
- So are there any grammars that generate languages that are not regular?
- For that matter, do any non-regular languages exist?
- Answers to these in the next chapter