Chapter Thirteen:
Stack Machines

Stacks are ubiquitous in computer programming, and they have an important role in formal language as well. A stack machine is a kind of automaton that uses a stack for auxiliary data storage. The size of the stack is unbounded—it never runs out of space—and that gives stack machines an edge over finite automata. In effect, stack machines have infinite memory, though they must use it in stack order.

If you travel by two paths that seem to depart in different directions, it is a surprise to discover that they lead to the same destination. It makes that destination feel more important—an intersection rather than a dead end. That is the situation with context-free languages. Stack machines and CFGs seem like two very different mechanisms for language definition—two paths that depart in different directions. But it turns out that these two paths lead to the same place. The set of languages that can be defined using a stack machine is exactly the same as the set of languages that can be defined using a CFG: the context-free languages.

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- 13.2 A Stack Machine for $\{a^n b^n\}$
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- 13.5 Example: Equal Counts
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Stacks

- A stack machine maintains an unbounded stack of symbols
- We'll represent these stacks as strings
- Left end of the string is the top of the stack
  - For example, *abc* is a stack with *a* on top and *c* on the bottom
  - Popping *abc* gives you the symbol *a*, leaving *bc* on the stack
  - Pushing *b* onto *abc* produces the stack *babc*

Stack Machine Moves

- A stack machine is an automaton for defining languages, but unlike DFA and NFA: no states!
- It is specified by a table that shows the moves it is allowed to make. For example:

read	pop	push
a	*c*	*a b c*

- Meaning:
  - If the current input symbol is *a*, and
  - if the symbol on top of the stack is *c*, it may make this move:
  - pop off the *c*, push *abc*, and advance to the next input symbol

Leaving The Stack Unchanged

- Every move pops one symbol off, then pushes a string of zero or more symbols on
- To specify a move that leaves the stack unchanged, you can explicitly push the popped symbol back on:

read	pop	push
a	*c*	*c*

- Meaning:
  - If the current input symbol is *a*, and
  - if the symbol on top of the stack is *c*, it may make this move:
  - pop off the *c*, push it back on, and advance to the next input symbol

Popping The Stack

- Every move pushes a string onto the stack
- To specify a move that pops but does not push, you can explicitly push the empty string:

read	pop	push
a	*c*	ε

- Meaning:
  - If the current input symbol is *a*, and
  - if the symbol on top of the stack is *c*, it may make this move:
  - pop off the *c*, push nothing in its place, and advance to the next input symbol

Moves On No Input

- The first column can be ε
- Like a ε-transition in an NFA, this specifies a move that is made without reading an input symbol

read	pop	push
ε	*c*	*a b*

- Meaning:
  - Regardless of what the next input symbol (if any) is,
  - if the symbol on top of the stack is *c*, it may make this move:
  - pop off the *c*, and push *ab* in its place

Stack Machines

- A stack machine starts with a stack that contains just one symbol, the start symbol *S*
- On each move it can alter its stack, but only as we have seen: only in stack order
- Like an NFA, a stack machine may be nondeterministic: it may have more than one sequence of legal moves on a given input
- A string is in the language if there is at least one sequence of legal moves that reads the entire input string and ends with the stack empty

Example

	read	pop	push
1.	ε	*S*	*a b*
2.	*a*	*S*	*e f*
3.	*a*	*S*	ε

- Consider input *a* (and, as always, initial stack *S*):
- Three possible sequences of moves
  - Move 1 first: no input is read and the stack becomes *ab*; then stuck, rejecting since input not finished and stack not empty
  - Move 2 first: *a* is read and the stack becomes *ef*; rejecting since stack not empty
  - Move 3 first: *a* is read and the stack becomes empty; accepting

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Strategy For $\{a^n b^n\}$

- We'll make a stack machine that defines the language $\{a^n b^n\}$
- As always, the stack starts with *S*
- Reading the input string from left to right:
  1. For each *a* you read, pop off the *S*, push a 1, then push the *S* back on top
  – In the middle of the string, pop off the *S*; at this point the stack contains just a list of zero or more 1s, one for each *a* that was read
  – For each *b* you read, pop a 1 off the stack
- This ends with all input read and the stack empty, if and only if the input was in $\{a^n b^n\}$

Stack Machine For $\{a^n b^n\}$

	read	pop	push
1.	*a*	*S*	*S* 1
2.	ε	*S*	ε
3.	*b*	1	ε

- That strategy again:
  1. For each *a* you read, pop off the *S*, push a 1, then push the *S* back on top
  2. In the middle of the string, pop off the *S*; at this point the stack contains just a list of zero or more 1s, one for each *a* that was read
  3. For each *b* you read, pop a 1 off the stack

	read	pop	push
1.	*a*	*S*	*S* 1
2.	ε	*S*	ε
3.	*b*	1	ε

- Accepting *aaabbb*:
  - Start: input: *aaabbb*; stack: *S*
  - Move 1: input: *aaabbb*; stack: *S*1
  - Move 1: input: *aaabbb*; stack: *S*11
  - Move 1: input: *aaabbb*; stack: *S*111
  - Move 2: input: *aaabbb*; stack: 111
  - Move 3: input: *aaabbb*; stack: 11
  - Move 3: input: *aaabbb*; stack: 1
  - Move 3: input: *aaabbb_*; stack empty

	read	pop	push
1.	*a*	*S*	*S* 1
2.	ε	*S*	ε
3.	*b*	1	ε

- A rejecting sequence for *aaabbb*:
  - Start: input: *a*aabbb; stack: S
  - Move 1: input: *aa*abbb; stack: S1
  - Move 2: input: *aa*abbb; stack: 1
  - No legal move from here
- But, as we've seen, there is an accepting sequence, so *aaabbb* is in the language defined by the stack machine

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Strategy For $\{xx^R\}$

- We'll make a stack machine for $\{xx^R \mid x \in \{a,b\}^*\}$
- Reading the input string from left to right:
  1. For each *a* you read, pop off the *S*, push *a*, then push the *S* back on top
  – For each *b* you read, pop off the *S*, push *b*, then push the *S* back on top
  1. In the middle of the string, pop off the *S*; at this point the stack contains just a sequence of *a*s and *b*s, the reverse of the input string read so far
  – For each *a* you read, pop *a* off the stack
  – For each *b* you read, pop *b* off the stack
- This ends with all input read and the stack empty, if and only if the input was in $\{xx^R \mid x \in \{a,b\}^*\}$

Stack Machine For $\{xx^R\}$

	read	pop	push
1.	*a*	*S*	*S a*
2.	*b*	*S*	*S b*
3.	ε	*S*	ε
4.	*a*	*a*	ε
5.	*b*	*b*	ε

- That strategy again:
  1. For each *a* you read, pop off the *S*, push *a*, then push the *S* back on top
  2. For each *b* you read, pop off the *S*, push *b*, then push the *S* back on top
  3. In the middle of the string, pop off the *S*; at this point the stack contains just a sequence of *a*s and *b*s, the reverse of the input string read so far
  4. For each *a* you read, pop *a* off the stack
  5. For each *b* you read, pop *b* off the stack

	read	pop	push
1.	*a*	*S*	*S a*
2.	*b*	*S*	*S b*
3.	ε	*S*	ε
4.	*a*	*a*	ε
5.	*b*	*b*	ε

- Accepting *abbbba*:
  - Start: input: *abbbba*; stack: *S*
  - Move 1: input: *abbbba*; stack: *Sa*
  - Move 2: input: *abbbba*; stack: *Sba*
  - Move 2: input: *abbbba*; stack: *Sbba*
  - Move 3: input: *abbbba*; stack: *bba*
  - Move 5: input: *abbbba*; stack: *ba*
  - Move 5: input: *abbbba*; stack: *a*
  - Move 4: input: *abbbba_*; stack empty

Nondeterminism

- This stack machine (like the previous) can pop the S off the top of the stack at any time
- But there is only one correct time: it must be popped off in the middle of the input string
- This uses the nondeterminism of stack machines
- We can think of these machines as making a guess about where the middle of the input is
- All the sequences with a wrong guess reject
- But the one sequence that makes the right guess accepts, and one is all it takes

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The 4-Tuple

- A *stack machine M* is a 4-tuple $M = (\Gamma, \Sigma, S, \delta)$
  - Γ is the stack alphabet
  - Σ is the input alphabet
  - $S \in \Gamma$ is the initial stack symbol
  - $\delta \in ((\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow P(\Gamma^*)$ is the transition function
- The stack alphabet and the input alphabet may or may not have symbols in common

Transition Function

- Type is $\delta \in ((\Sigma \cup \{\varepsilon\}) \times \Gamma \rightarrow P(\Gamma^*)$
- That is, in $\delta(x,y) = Z$:
  - x is an input symbol or ε
  - y is a stack symbol
  - The result Z is a set of strings of stack symbols
- The result is a set because the stack machine is nondeterministic
- For a given input symbol x and top-of-stack symbol y, there may be more than one move
- So, there may be more than one string that can be pushed onto the stack in place of y

Example

	read	pop	push
1.	ε	S	$a\,b$
2.	a	S	$e\,f$
3.	a	S	ε

- $M = (\Gamma, \Sigma, S, \delta)$ where
  - $\Gamma = \{S, a, b, e, f\}$
  - $\Sigma = \{a\}$
  - $\delta(\varepsilon, S) = \{ab\}$
    $\delta(a, S) = \{\varepsilon, ef\}$

Instantaneous Descriptions

- At any point in a stack machine's operation, its future depends on two things:
  - That part of the input string that is still to be read
  - The current contents of the stack
- An instantaneous description (ID) for a stack machine is a pair (*x*, *y*) where:
  - $x \in \Sigma^*$ is the unread part of the input
  - $y \in \Gamma^*$ is the current stack contents
- As always, the left end of the string *y* is considered to be the top of the stack

A One-Move Relation On IDs

- We will write $I \mapsto J$ if I is an ID and J is ID that follows from I after one move of the stack machine
- Technically: $\mapsto$ is a relation on IDs, defined by the δ function for the stack machine as follows:
  - Regular transitions: $(ax, Bz) \mapsto (x, yz)$ if and only if $y \in \delta(a,B)$
  - ε-transitions: $(x, Bz) \mapsto (x, yz)$ if and only if $y \in \delta(\varepsilon,B)$.
- Note no move is possible when stack is empty

Zero-Or-More-Move Relation

- As we did with grammars and NFAs, we extend this to a zero-or-more-move $\mapsto^*$
- Technically, $\mapsto^*$ is a relation on IDs, with $I \mapsto^* J$ if and only if there is a sequence of zero or more relations that starts with I and ends with J
- Note this is reflexive by definition: we always have $I \mapsto^* I$ by a sequence of zero moves

A Stack Machine's Language

- The language accepted by a stack machine is the set of input strings for which there is at least one sequence of moves that ends with the whole string read and the stack empty
- Technically, $L(M) = \{x \in \Sigma^* \mid (x, S) \mapsto^* (\varepsilon, \varepsilon)\}$

Previous Example

	read	pop	push
1.	*a*	*S*	*S a*
2.	*b*	*S*	*S b*
3.	ε	*S*	ε
4.	*a*	*a*	ε
5.	*b*	*b*	ε

- Accepting abbbba:
  - Start: input: abbbba; stack: S
  - Move 1: input: abbbba; stack: Sa
  - Move 2: input: abbbba; stack: Sba
  - Move 2: input: abbbba; stack: Sbba
  - Move 3: input: abbbba; stack: bba
  - Move 5: input: abbbba; stack: ba
  - Move 5: input: abbbba; stack: a
  - Move 4: input: abbbba_; stack empty

Example, Continued

	read	pop	push
1.	a	S	$S\,a$
2.	b	S	$S\,b$
3.	ε	S	ε
4.	a	a	ε
5.	b	b	ε

- $M = (\{a,b,S\}, \{a,b\}, S, \delta)$, where
  - $\delta(a,S) = \{Sa\}$ $\quad \delta(b,S) = \{Sb\}$ $\quad \delta(\varepsilon,S) = \{\varepsilon\}$
    $\delta(a,a) = \{\varepsilon\}$ $\quad \delta(b,b) = \{\varepsilon\}$
- The accepting sequence of moves for *abbbba* is
  - $(abbbba, S) \mapsto (bbbba, Sa) \mapsto (bbba, Sba) \mapsto (bba, Sbba)$
    $\mapsto (bba, bba) \mapsto (ba, ba) \mapsto (a, a) \mapsto (\varepsilon, \varepsilon)$
- $(abbbba, S) \mapsto^* (\varepsilon, \varepsilon)$ and so $abbbba \in L(M)$

Considering $L(M)$

	read	pop	push
1.	a	S	$S\,a$
2.	b	S	$S\,b$
3.	ε	S	ε
4.	a	a	ε
5.	b	b	ε

- Only move 3 changes the number of S on the stack
- So any accepting sequence uses 3 exactly once:

$$(xy, S) \mapsto^* (y, Sz) \mapsto_3 (y, z) \mapsto^* (\varepsilon, \varepsilon)$$

  with moves 1 and 2 before 3, and 4 and 5 after
- 1 and 2 push the symbol just read, so $z = x^R$
- 4 and 5 pop symbols matching input, so $z = y$
- So accepting sequences are all of this form:

$$(xx^R, S) \mapsto^* (x^R, Sx^R) \mapsto_3 (x^R, x^R) \mapsto^* (\varepsilon, \varepsilon)$$

- $L(M) = \{xx^R \mid x \in \{a,b\}^*\}$

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A Language Of Equal Counts

- Let $A = \{x \in \{a,b\}^* \mid$ the number of as in x equals the number of bs in $x\}$
- Not the same as $\{a^n b^n\}$, since $abab \in A$
- A first attempt:
  - For each a, push a 0
  - For each b, pop a 0
  - Pop off the S (at the end)

First Attempt

	read	pop	push
1.	*a*	*S*	0 *S*
2.	*a*	0	0 0
3.	*b*	0	ε
4.	ε	*S*	ε

- Moves 1 and 2 push a 0 for each *a*
- Move 3 pops a 0 for each *b*
- Move 4 pops the *S*, leaving the stack empty
- In any accepting sequence, move 4 comes last:
  - It's the only way to move the *S*
  - Once the stack is empty, no further moves are possible

First Attempt

	read	pop	push
1.	*a*	*S*	0 *S*
2.	*a*	0	0 0
3.	*b*	0	ε
4.	ε	*S*	ε

- $M = (\{0,S\}, \{a,b\}, S, \delta)$, where
  - $\delta(a,S) = \{0S\}$ $\quad\quad$ $\delta(a,0) = \{00\}$
    $\delta(b,0) = \{\varepsilon\}$ $\quad\quad$ $\delta(\varepsilon,S) = \{\varepsilon\}$
- The accepting sequence of moves for *abab* is
  - $(abab, S) \mapsto (bab, 0S) \mapsto (ab, S) \mapsto (b, 0S) \mapsto (\varepsilon, S) \mapsto (\varepsilon, \varepsilon)$
- $(abab, S) \mapsto^* (\varepsilon, \varepsilon)$ and so $abab \in L(M)$

Problem

- This works on some examples, but not all
- Everything *M* accepts is in our target language *A*, but not everything in *A* is accepted by *M*
- Consider *abba:* $(abba, S) \mapsto (bba, 0S) \mapsto (ba, S) \mapsto ?$
- *M* has no way to accept strings like *abba* that have a prefix with more *b*s than *a*s

Improved Strategy

- When the number of *a*s so far exceeds the number of *b*s, keep count with 0s on the stack, as before
- When the number of *b*s so far exceeds the number of *a*s, keep count with 1s instead
- At the end, pop off the *S*

Solution

- $M = (\{0,S\}, \{a,b\}, S, \delta)$, where
  - $\delta(a,S) = \{0S\}$ $\quad \delta(a,0) = \{00\}$
    $\delta(b,0) = \{\varepsilon\}$ $\quad \delta(b,S) = \{1S\}$
    $\delta(b,1) = \{11\}$ $\quad \delta(a,1) = \{\varepsilon\}$
    $\delta(\varepsilon,S) = \{\varepsilon\}$

	read	pop	push
1.	*a*	*S*	0 *S*
2.	*a*	0	0 0
3.	*b*	0	ε
4.	*b*	*S*	1 *S*
5.	*b*	1	1 1
6.	*a*	1	ε
7.	ε	*S*	ε

- The accepting sequence of moves for *abba* is
  - $(abba, S) \mapsto (bba, 0S) \mapsto (ba, S) \mapsto (a, 1S) \mapsto (\varepsilon, S) \mapsto (\varepsilon, \varepsilon)$
- $(abba, S) \mapsto^* (\varepsilon, \varepsilon)$ and so $abba \in L(M)$
- This solution now has $L(M) = A$

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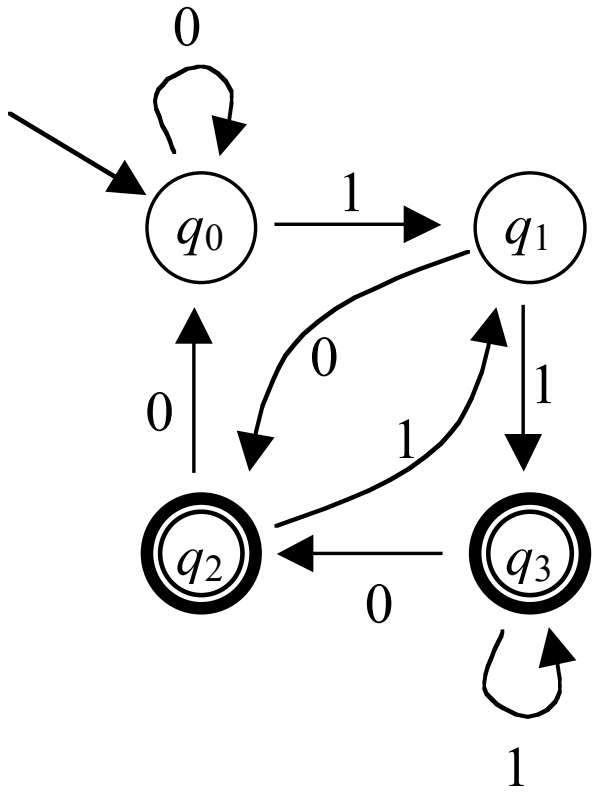
Simulating DFAs

- A stack machine can easily simulate any DFA
  - Use the same input alphabet
  - Use the states as stack symbols
  - Use the start state as the start symbol
  - Use a transition function that keeps exactly one symbol on the stack: the DFA's current state
  - Allow accepting states to be popped; that way, if the DFA ends in an accepting state, the stack machine can end with an empty stack

Example



- $M = (\{q_0, q_1, q_2, q_3\}, \{0,1\}, q_0, \delta)$
  - $\delta(0,q_0) = \{q_0\}$ $\quad\quad$ $\delta(1,q_0) = \{q_1\}$
  - $\delta(0,q_1) = \{q_2\}$ $\quad\quad$ $\delta(1,q_1) = \{q_3\}$
  - $\delta(0,q_2) = \{q_0\}$ $\quad\quad$ $\delta(1,q_2) = \{q_1\}$
  - $\delta(0,q_3) = \{q_2\}$ $\quad\quad$ $\delta(1,q_3) = \{q_3\}$
  - $\delta(\varepsilon,q_2) = \{\varepsilon\}$ $\quad\quad$ $\delta(\varepsilon,q_3) = \{\varepsilon\}$
- Accepting sequence for 0110:
  - $(0110, q_0) \mapsto (110, q_0) \mapsto (10, q_1) \mapsto (0, q_3) \mapsto (\varepsilon, q_2) \mapsto (\varepsilon, \varepsilon)$

DFA To Stack Machine

- Such a construction can be used to make a stack machine equivalent to any DFA
- It can be done for NFAs too
- It tells us that the languages definable using a stack machine include, at least, all the regular languages
- In fact, regular languages are a snap: we have an unbounded stack we barely used
- We won't give the construction formally, because we can do better…

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From CFG To Stack Machine

- A CFG defines a string rewriting process
- Start with *S* and rewrite repeatedly, following the rules of the grammar until fully terminal
- We want a stack machine that accepts exactly those strings that could be generated by the given CFG
- Our strategy for such a stack machine:
  - Do a derivation, with the string in the stack
  - Match the derived string against the input

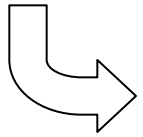
Strategy

- Two types of moves:
  1. A move for each production $X \rightarrow y$
  2. A move for each terminal $a \in \Sigma$
- The first type lets it do any derivation
- The second matches the derived string and the input
- Their execution is interlaced:
  - type 1 when the top symbol is nonterminal
  - type 2 when the top symbol is terminal

read	pop	push
ε	X	y
a	a	ε

Example: $\{xx^R \mid x \in \{a,b\}^*\}$

$S \rightarrow aSa \mid bSb \mid \varepsilon$



	read	pop	push
1.	ε	S	aSa
2.	ε	S	bSb
3.	ε	S	ε
4.	a	a	ε
5.	b	b	ε

- Derivation for *abbbba:*

  $S \Rightarrow aSb \Rightarrow abSba \Rightarrow abbSbba \Rightarrow abbbba$

- Accepting sequence of moves on *abbbba:*

  $(abbbba, S) \mapsto_1 (abbbba, aSa) \mapsto_4 (bbbba, Sa) \mapsto_2 (bbbba, bSba) \mapsto_5$
  $(bbba, Sba) \mapsto_2 (bbba, bSbba) \mapsto_5 (bba, Sbba) \mapsto_3 (bba, bba) \mapsto_5$
  $(ba, ba) \mapsto_5 (a, a) \mapsto_4 (\varepsilon, \varepsilon)$

Lemma 13.7

If $G = (V, \Sigma, S, P)$ is any context-free grammar, there is some stack machine M with $L(M) = L(G)$.

- Proof sketch: by construction
- Construct $M = (V \cup \Sigma, \Sigma, S, \delta)$, where
  - for all $v \in V$, $\delta(\varepsilon, v) = \{x \mid (v \rightarrow x) \in P\}$
  - for all $a \in \Sigma$, $\delta(a, a) = \{\varepsilon\}$
- M accepts x if and only if G generates x
- $L(M) = L(G)$
- (Missing: detailed proof that $(x, S) \mapsto^* (\varepsilon, \varepsilon)$ if and only if $S \Rightarrow^* x$)

Summary

- We can make a stack machine for every CFL
- That's stronger than our demonstration in the last chapter of a stack machine for every regular language
- So now we know that the stack machines are at least as powerful as CFGs for defining languages
- Are they more powerful? Are there stack machines that define languages that are not CFLs?

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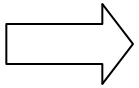
From Stack Machine To CFG

- We can't just reverse the previous construction, since it produced restricted productions
- But we can use a similar idea
- The executions of the stack machine will be exactly simulated by derivations in the CFG
- To do this, we'll construct a CFG with one production for each move of the stack machine

	read	pop	push
1.	*a*	*S*	0 *S*
2.	*a*	0	0 0
3.	*b*	0	ε
4.	*b*	*S*	1 *S*
5.	*b*	1	1 1
6.	*a*	1	ε
7.	ε	*S*	ε



1. $S \rightarrow a0S$
2. $0 \rightarrow a00$
3. $0 \rightarrow b$
4. $S \rightarrow b1S$
5. $1 \rightarrow b11$
6. $1 \rightarrow a$
7. $S \rightarrow \varepsilon$

- One-to-one correspondence:
  - Where the stack machine has $t \in \delta(\omega, A)$…
  - … the grammar has $A \rightarrow \omega t$
- Accepting sequence on *abab:*

  $(abab, S) \mapsto_1 (bab, 0S) \mapsto_3 (ab, S) \mapsto_1 (b, 0S) \mapsto_3 (\varepsilon, S) \mapsto_7 (\varepsilon, \varepsilon)$
- Derivation of *abab:*

  $S \Rightarrow_1 a0S \Rightarrow_3 abS \Rightarrow_1 aba0S \Rightarrow_3 ababS \Rightarrow_7 abab$

Lemma 13.8.1

If $M = (\Gamma, \Sigma, S, \delta)$ is any stack machine, there is context-free grammar G with $L(G) = L(M)$.

- Proof by construction
- Assume that $\Gamma \cap \Sigma = \{\}$ (without loss of generality)
- Construct $G = (\Gamma, \Sigma, S, P)$, where
  $P = \{(A \rightarrow \omega t) \mid A \in \Gamma, \omega \in \Sigma \cup \{\varepsilon\}, \text{ and } t \in \delta(\omega, A)\}$
- Now leftmost derivations in G simulate runs of M:
  $S \Rightarrow^* xy$ if and only if $(x, S) \mapsto^* (\varepsilon, y)$
  for any $x \in \Sigma^*$ and $y \in \Gamma^*$ (see the next lemma)
- Setting $y = \varepsilon$, this gives
  $S \Rightarrow^* x$ if and only if $(x, S) \mapsto^* (\varepsilon, \varepsilon)$
- So $L(G) = L(M)$

Disjoint Alphabets Assumption

- The stack symbols of the stack machine become nonterminals in the CFG
- The input symbols of the stack machine become terminals of the CFG
- That's why we need to assume $\Gamma \cap \Sigma = \{\}$: symbols in a grammar must be either terminal or nonterminal, not both
- This assumption is without loss of generality because we can easily rename stack machine symbols to get disjoint alphabets…

Renaming Example

- Given a stack machine with intersecting alphabets:

	read	pop	push
1.	*a*	*S*	*S b b*
2.	ε	*S*	ε
3.	*b*	*b*	ε

- We can rename the stack symbols (the pop and push columns only) to get disjoint alphabets:

	read	pop	push
1.	*a*	*S*	*S B B*
2.	ε	*S*	ε
3.	*b*	*B*	ε

- Then use the construction:

$$S \rightarrow aSBB \mid \varepsilon$$
$$B \rightarrow b$$

Missing Lemma

- Our proof claimed that the leftmost derivations in *G* exactly simulate the executions of *M*
- Technically, our proof said:

$$S \Rightarrow^* xy \text{ if and only if } (x,S) \mapsto^* (\varepsilon,y)$$
$$\text{for any } x \in \Sigma^* \text{ and } y \in \Gamma^*$$

- This assertion can be proved in detail using induction
- We have avoided most such proofs this far, but this time we'll bite the bullet and fill in the details

Lemma 13.8.2

For the construction of the proof of Lemma 13.8.1, for any $x \in \Sigma^*$ and $y \in \Gamma^*$, $S \Rightarrow^* xy$ if and only if $(x,S) \mapsto^* (\varepsilon,y)$.

- Proof that if $S \Rightarrow^* xy$ then $(x,S) \mapsto^* (\varepsilon,y)$ is by induction on the length of the derivation
- Base case: length is zero
  - $S \Rightarrow^* xy$ with $xy = S$; since $x \in \Sigma^*$, $x = \varepsilon$ and $y = S$
  - For these values $(x,S) \mapsto^* (\varepsilon,y)$ in zero steps
- Inductive case: length greater than zero
  - Consider the corresponding leftmost derivation $S \Rightarrow^* xy$
  - In G, leftmost derivations always produce a string of terminals followed by a string of nonterminals
  - So we have $S \Rightarrow^* x'Ay' \Rightarrow xy$, for some $x' \in \Sigma^*$, $A \in \Gamma$, and $y' \in \Gamma^*$…

Lemma 13.8.2, Continued

For the construction of the proof of Lemma 13.8.1, for any $x \in \Sigma^*$ and $y \in \Gamma^*$, $S \Rightarrow^* xy$ if and only if $(x,S) \mapsto^* (\varepsilon,y)$.

- Inductive case, continued:
  - $S \Rightarrow^* x'Ay' \Rightarrow xy,$ for some $x' \in \Sigma^*$, $A \in \Gamma$, and $y' \in \Gamma^*$
  - By the inductive hypothesis, $(x',S) \mapsto^* (\varepsilon,Ay')$
  - The final step uses one of the productions $(A \rightarrow \omega t)$
  - So $S \Rightarrow^* x'Ay' \Rightarrow x'\omega ty' = xy,$ where $x'\omega = x$ and $ty' = y$
  - Since $(x',S) \mapsto^* (\varepsilon,Ay')$, we also have $(x'\omega,S) \mapsto^* (\omega,Ay')$
  - For production $(A \rightarrow \omega t)$ there must be a move $t \in \delta(\omega,A)$
  - Using this as the last move,
    $(x,S) = (x'\omega,S) \mapsto^* (\omega,Ay') \mapsto (\varepsilon,ty') = (\varepsilon,y)$
  - So $(x,S) \mapsto^* (\varepsilon,y)$, as required

Lemma 13.8.2, Continued

For the construction of the proof of Lemma 13.8.1, for any $x \in \Sigma^*$ and $y \in \Gamma^*$, $S \Rightarrow^* xy$ if and only if $(x,S) \mapsto^* (\varepsilon,y)$.

- We have shown one direction of the "if and only if":
  - if $S \Rightarrow^* xy$ then $(x,S) \mapsto^* (\varepsilon,y)$

  by induction on the number of steps in the derivation
- It remains to show the other direction:
  - if $(x,S) \mapsto^* (\varepsilon,y)$ then $S \Rightarrow^* xy$
- The proof is similar, by induction on the number of steps in the execution

Theorem 13.8

A language is context free if and only if it is $L(M)$ for some stack machine M.

- Proof: follows immediately from Lemmas 13.7 and 13.8.1.
- Conclusion: CFGs and stack machines have equivalent definitional power