SE 504 (Formal Methods and Models)
The strength/weakness relationship between predicates

Recall that a *predicate* is simply a function that yields a boolean value and that [.] is the *everywhere* operator on predicates; i.e., for a predicate P, the expression $[P]$ is true if P holds in all states (i.e., everywhere) but false if there is at least one state in which P does not hold. Technically,

$$[P] \ \hat{=} \ (\forall x_1, x_2, \ldots, x_n \mid : \ P)$$

where the x_i's are precisely those variables that occur free in P.

Let P and Q be predicates. Then, with respect to weakness/strength, the possible relationships between them are as follows:

If $[P \Rightarrow Q]$ (equivalently, $[Q \Leftarrow P]$), we say that P is *stronger than* Q (equivalently, Q is *weaker than* P).

If each of P and Q is stronger than the other, we say that they *are equivalent*. This makes sense, because

$$[P \Rightarrow Q] \wedge [P \Leftarrow Q] \ \equiv \ [P \equiv Q]$$

If, on the other hand, P is stronger than Q (equivalently, Q is weaker than P), but Q is not stronger than P (equivalently, P is not weaker than Q), we say that P is *strictly stronger than* Q (equivalently, Q is *strictly* weaker than P).

If neither P is stronger than Q nor Q is stronger than P, then P and Q are unrelated with respect to weakness/strength.

These are summarized in the following table:

$[P \Rightarrow Q]$	$[P \Leftarrow Q]$	Relationship
true	true	P and Q are equivalent
false	true	P is strictly weaker than Q
true	false	P is strictly stronger than Q
false	false	P and Q are unrelated

In order to demonstrate that $[P \Rightarrow Q]$ is false, it suffices to identify a state in which $P \Rightarrow Q$ is false (i.e., a state that satisfies P but fails to satisfy Q).

In order to demonstrate that $[P \Rightarrow Q]$ is true, it suffices to prove $P \Rightarrow Q$. See Metatheorem 9.16 (and the accompanying discussion) in the text by Gries and Schneider. Such a proof can be of the form taught in the aforementioned text, or it could be a little less formal. One could, for example, consider an arbitrary state s satisfying P and argue persuasively that s also satisfies Q.

Note: An "arbitrary" state is one about which nothing is assumed; students often make the mistake of choosing a particular state (having properties convenient to their purposes) and calling it "arbitrary". **End of note.**

As an example, suppose we have $P : x \geq 1 \wedge y < x$ and $Q : x \geq 0$, with x and y of type `integer`. Then P is strictly stronger than Q, because $[P \Rightarrow Q]$ holds but $[P \Leftarrow Q]$ does not. The latter follows from $P \Leftarrow Q$ being false in any state in which $x = 0$. The former is rather obvious, but may be proved rigorously as follows:

$$x \geq 0$$

$= \quad \langle$ integer arithmetic $\rangle$

$$x = 0 \ \vee \ x \geq 1$$

$\Leftarrow \quad \langle\ p \Rightarrow p \vee q$ (Gries 3.76a) $\rangle$

$$x \geq 1$$

$\Leftarrow \quad \langle\ p \wedge q \ \Rightarrow \ p$ (Gries 3.76b) $\rangle$

$$x \geq 1 \ \wedge \ y < x$$