Let $S$ be a program and $Q$ be a predicate (over the state space of $S$). The expression $\text{wp}.S.Q$ (read “weakest precondition of $S$ with respect to $Q$”) refers to the weakest predicate $P$ satisfying the Hoare triple $\{P\} S \{Q\}$. In other words

$$\{P\} S \{Q\} \equiv [P \Rightarrow \text{wp}.S.Q]$$

Among the laws pertaining to $\text{wp}$ are these:

- **wp skip law:** $[\text{wp}.\text{skip}.Q \equiv Q]$
- **wp assignment law:** $[\text{wp}.(x := E).Q \equiv Q(x := E)]$
- **wp catenation law:** $[\text{wp}.(S_1; S_2).Q \equiv \text{wp}.S_1.(\text{wp}.S_2.Q)]$

The wp catenation law says, in effect, that the weakest solution to $\{?\} S_1; S_2 \{Q\}$ is none other than $\text{wp}.S_1.R$ (i.e., the weakest solution to $\{?\} S_1 \{R\}$), where $R$ is $\text{wp}.S_2.Q$ (i.e., the weakest solution to $\{?\} S_2 \{Q\}$).

That is, to obtain the weakest precondition for the catenation $S_1; S_2$ (with respect to a postcondition $Q$), we first find the weakest precondition for $S_2$ (with respect to $Q$), which serves as our postcondition for $S_1$.

In problems 1-3, simplify the given expression as much as possible. Use the wp laws given above, as well as well-known theorems from arithmetic, algebra, and logic. Regarding Problem 2, note that catenation is associative, meaning that $(S_1; S_2); S_3$ and $S_1; (S_2; S_3)$ are equivalent programs. Problem 3, despite being worded differently, is the same kind of problem as the ones preceding it.

1. $\text{wp}.(i := i + 2 \ast j; \ j := i - j).(i < 2j)$
2. $\text{wp}.(y := x - y; \ x := x - y; \ y := y + x).(x = Y \land y = X)$
3. Determine the weakest predicate $P$ that makes this Hoare Triple true:

$$\{P\} k := k + 1; \ sum := sum + b.(k - 1) \{\text{sum} = (+j \mid i \leq j < k : b.j) \land i < k \leq \#b\}$$
The remaining problems involve Hoare Triples whose programs include a selection command and a catenation of commands.

Recall that if $\text{IF } P \text{ THEN IF } Q$ is the program

\[
\text{IF } (P \Rightarrow (B_0 \lor B_1)) \land (P \land B_0) \{Q\} \land (P \land B_1) \{Q\}
\]

which generalizes in the natural way when $\text{IF}$ has more than two branches.

4. Prove

\[
\{P \land i < \#b\}
\]

\[
\text{if } b.i \neq 0 \rightarrow \text{prod} := \text{prod} \ast b.i; \ i := i + 1
\]

\[
\text{fi}
\]

\[
\{P \land i \leq \#b\}
\]

where $P : 0 \leq i \land \text{prod} = \bigoplus_{j : 0 \leq j < i} b.j$ $\neq 0 : b.j$

Notice that the first branch of the selection command is a catenation of two assignment commands. Thus, in showing that that branch behaves as intended, you must make use of a catenation law.

Hint 1: A quantification range such as $0 \leq i < n + 1 \land R$ can be rewritten as the disjunction $(0 \leq i < n \land R) \lor (i = n \land R)$ (first by rewriting $0 \leq i < n + 1$ as $0 \leq i < n \lor i = n$ and then by applying (3.46)), after which Range Split (8.16) is applicable.

Hint 2: A quantification range of the form $P \land R$, where $R$ does not mention a dummy, can, in some circumstances, be simplified either to $P$ or to false, the former when $R$ can be reduced to true and the latter when $R$ can be reduced to false.

Hint 3: Theorem (3.84a) tells us that the conjunction $(e = f) \land P$ is equivalent to $(e = f) \land P'$, where $P'$ is obtained from $P$ by replacing one or more occurrences of $e$ by $f$. If $e$ is a dummy and $f$ is not, this is one way of getting rid of a dummy in a conjunct. (See Hint 2.)

5. Prove

\[
\{P \land 0 \leq k < \#b\}
\]

\[
\text{if } b.k \leq 0 \rightarrow \text{sum} := \text{sum} - b.k
\]

\[
\text{fi}
\]

\[
\{P\}
\]

where $P : \text{sum} = (+i \mid 0 \leq i < k : |b.i|)$ and where $|x|$ is the absolute value of $x$, defined as follows:

\[
[(|x| = x \equiv x \geq 0) \land (|x| = -x \equiv x \leq 0)]
\]

Notice that the program is a catenation of a selection command and an assignment command. Thus, to show that the Hoare Triple is valid you should make use of a catenation law to provide the appropriate postcondition for the selection command.